Business Mathematics

notes and projections from lecture

Kit Tyabandha, PhD

God's Ayudhya's Defence
Bangkok

14^{th} January, 2007

Catalogue in Publication Data
Kit Tyabandha
Business Mathematics, notes and projections from lecture: – Bangkok, Kittix, GAD, 2005
323 p.
1. Business Mathematics   I. Tyabandha, Kit   II. Mathematics.
510
ISBN 974–94279–5–5

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Published by Kittix Publishing
God's Ȧyudhyạ's Defence
1564/11 Prajarastrasaya 1 Road
Bangkok 10800, Thailand

Editor
Vaen Sriwayudhya

Printed in Thailand by
Kittix Press

Typeset using TEX

God's Ȧyudhyạ's Defence
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Kittix

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To God.

Preface

Business Mathematics is a branch of applied mathematics that uses calculus, algebra and mathematical programming, to mention but a few. It finds applications in finance and economics. The mathematical analysis of stock markets, portfolios and the management of risks may also be covered by this subject.

I began working on this book in October 2005 for the course Business Mathematics that I taught at Mahidol University at Kancanaburi. This volume is the collection of hand-outs which were given to students before each lecture. Apart from examples and exercises, exam papers are also included. They are put in the appendix and represent in a way more examples.

These hand-outs were used hand in hand with another set of printed pages whose contents are in larger letter, which were used on the camera projector. It would be good if in the future we could compile these hand-outs into lecture notes to be published as a book.

There were 66 students in the class. From after the midterm exam my main concern has been for the students to do the exercises and problems by themselves. Since in practice this was hardly the case with the homework, we turned to holding within our class sessions of exercise practice and quiz. This has worked to a certain degree, all the time one main problem remain being the large size of the class, I do hope the students have learnt the importance of problem-oriented approach, and that each of them will find it useful in the future. I hope they have learnt from me as much as I have learnt for and from them.

Kit Tyabandha, PhD
Bangkok, 14th January, 2007

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Graph and derivative

25^{th} October 2005

Definition 1. Let n be a positive integer. Then x^n means that x is multiplied to itself n times.

§

Definition 2. A *monomial* is an expression cosisting a real-numbered coefficient times one or more variables each raised to the power of a positive integer. Adding and subtracting monomials to one another give us a *polynomial.* A *univariate polynomial* is a polynomial of one variable, of the form $a_n x^n + \ldots + a_1 x + a_0$, and its degree is the highest power of that variable, that is n. The degree of a polynomial is sometimes known as its *order.* A polynomial can be simplified into a product of two polynomials through the process called *factoring.* A *polynomial equation* is an equation of the form $p(\cdot) = 0$, where $p(\cdot)$ is a polynomial.

§

Definition 3. A *quadratic equation* is a second-degree polynomial equation, that is one of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.

§

Example 1. The factors of $mx^2 + nx + p$ are $ax + b$ and $cx + d$, where $ac = m$, $bd = p$ and $ad + bc = n$. Quadratic equations may be solved by factoring.

Example 2. A quadratic polynomial of the form $x^2 + bx$ can be transformed into the form $(x + a)^2$ by adding $(b/2)^2$ and then factor the result. *Completing the square* in the quadratic equation $ax^2 + bx + c = 0$ leads one to

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

which gives us the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition 4. At a point on the graph where the first derivative is zero, if the second derivative is positive, then that point is a *local minimum*; if the second derivative is negative, it is a *local maximum*; on the other hand, if the second derivative is either zero or is undefined, it is an *inflection point.* An *asymptote* is a straight line to which a non-linear curve smoothly approach as it goes towards infinity, never reaching it. At any point on a graph, if the first derivative is positive the function at that point is *increasing*, and if negative it is *decreasing.* At any point on a graph, if the second derivative is positive it is said that the function is *convex* at that point, and if negative the latter is said to be *concave*.

§

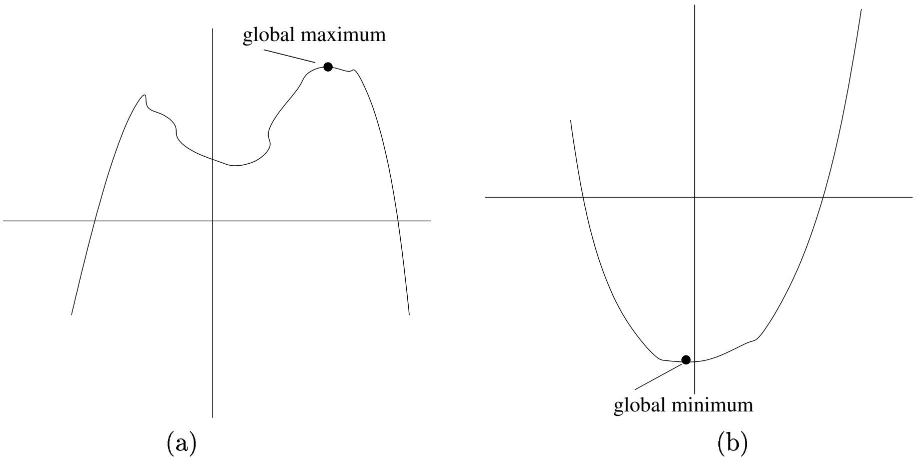


Figure 1 *(a) global maximum-, and (b) global minimum points of a function.*

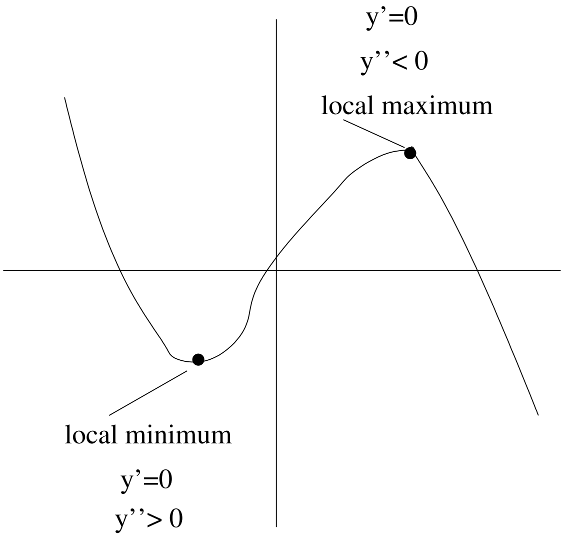
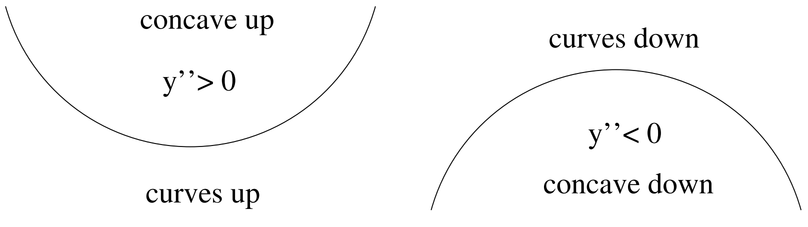
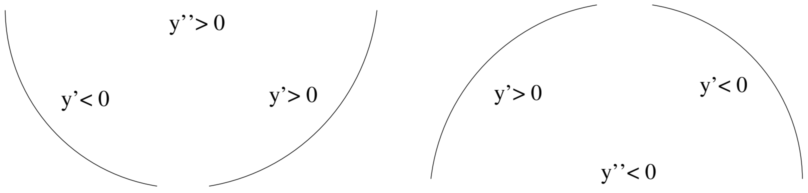


Figure 2 *Local minimum- and local maximum points of a function.*



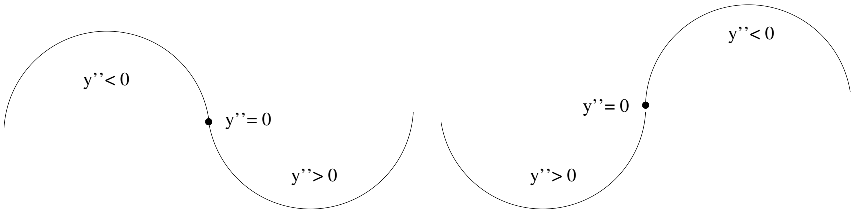
(a) (b)

Figure 3 *The two curvature types, namely (a) concave up ($y'' > 0$) and, (b) concave down ($y'' < 0$).*



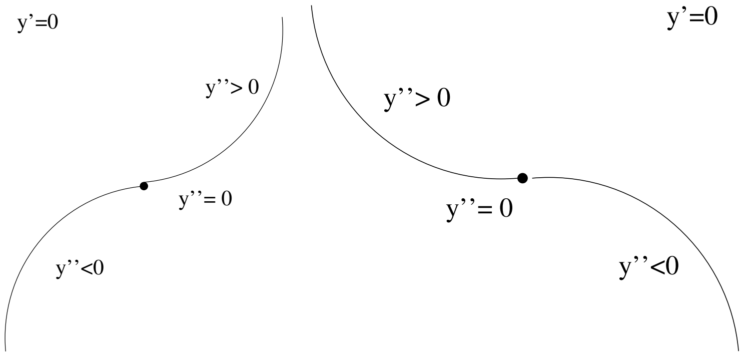
(a) (b)

Figure 4 *The four possible curvatures in two dimensions, considering both y' and y'', (a) $y'' > 0$ and, (b) $y'' < 0$.*



(a) (b)

Figure 5 *Inflection points, where $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*



(a) (b)

Figure 6 *Stationary inflection points, where both* $y' = 0$ *and* $y'' = 0$, *(a) with* y'' *increasing and, (b) with* y'' *decreasing.*

Example 3. The cubic function of the form $y = ax^3 + bx^2 + cx + d$ has local maximum and minimum, when these exist, at the point where the first derivative is zero, that is when $3ax^2 + 2bx + c = 0$. At such points,

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Depending on whether $b^2 - 3ac$ is positive or negative, the local minimum and maximum either exist or do not exist. As examples, when $a = 1$, $b = 2$, $c = 3$ and $d = 4$ this value is -5, and therefore the value of x becomes complex. In the space of real number this means that there is neither a maximum- or a minimum point. Instead, in this case we have an inflection point, which is the point where the second derivative becomes zero, that is to say, $6ax + 2b = 0$, or $x = -\frac{b}{3a}$. Figure 7 shows a plot of this case.

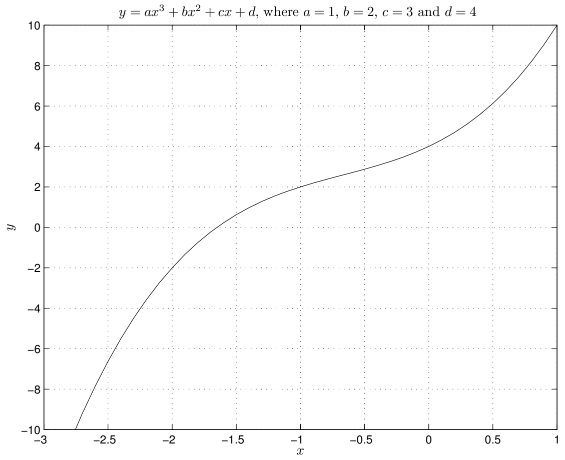


Figure 7 *The cubic function* $y = x^3 + 2x^2 + 3x + 4$.

Figure 8 shows the case where $a = 1$, $b = 5$, $c = 4$ and $d = 3$

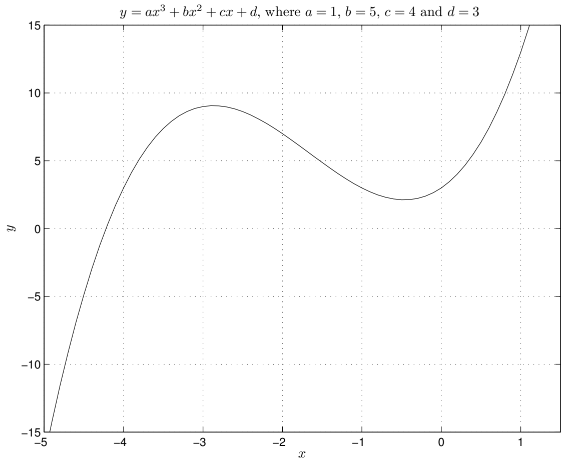


Figure 8 *The cubic function* $y = x^3 + 5x^2 + 4x + 3$.

Example 4. The hyperbolic function of the form $(x - a)(y - b) = c$ has as its asymptotes the lines $x = a$ and $y = b$. Figure 9 shows the graph when $a = 1$, $b = 2$ and $c = 3$. Here the asymptotes are $x = 1$ and $y = 2$.

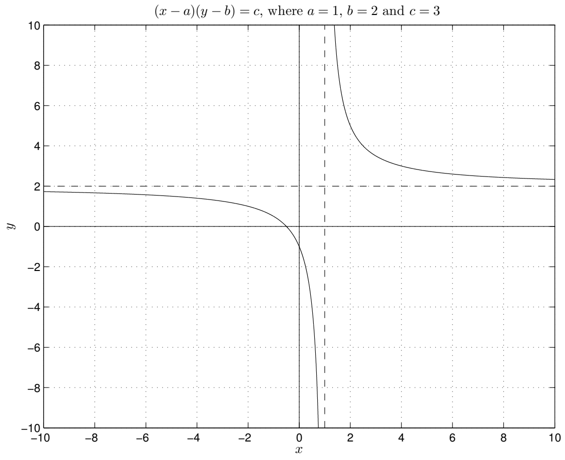


Figure 9 *The hyperbolic function* $(x - 1)(y - 2) = 3$.

Definition 5. The *straight line* has a form $ax + by + c = 0$, or $y = mx + d$ where $m = -a/b$ and $d = -c/b$. The y-intercept is the point where $x = 0$, that is $y = -c/b$ or $y = d$. The x-intercept is the point where $y = 0$, that is $x = -c/a$ or $x = -d/m$.

§

Example 5. Given a sum of money, an *isocost line* represents the different combinations of two inputs of production that can be bought. The general formula is $p_k k + p_l l = e$, where k and l are capital and labour, p_k and p_l their respective price, and e the allotted expenditures. In other words, $p_k k + p_l l - e = 0$, or $k = e/p_k - (p_l/p_k)l$. All the values concerned are positive, which means that the graph we are interested in is in the first quadrant only.

Example 6. At the simplest case where there are only two activities to choose from, if x and y are the two activites, then $t_x p_x + t_y p_y = m$, where t_x and t_y the number of hours spent on activities x and y respectively, p_x and p_y are the price per unit hour of x and y, and m is the limit of income.

Definition 6. The *general demand function* is of the form

$$q_d = \mathrm{f}(p, y, p_s, p_c, t_a, a, \ldots)$$

where q_d is the quantity demand of good x, p the price of x, y the income of the consumer, p_s the price of substitute goods, p_c the price of complementary goods, t_a the taste or fashion of the consumer, and a the advertisement level. In its simplest case where all other factors are constant, the *demand equation* takes the form $p = c_1 - c_2 q_d$, where p is the price-, while q_d the quantity demanded of the good x, and c_1 and c_2 are positive constants. The *general supply function* is of the form $q_s = \mathrm{f}(p, c, p_0, t_e, n, o, \ldots)$, where q_s is the quantity supplied of good x, p the price of x, c the cost of production, p_0 the price of other goods, t_e the available technology, n the number of producers in the market, and o other factors, for example tax and subsidies. The simplified relation for the *supply* is $p = c_1 + c_2 q_s$, where q_s is the quantity of x supplied, and c_1 and c_2 are positive constants.

§

Example 7. When a tax of t per unit is imposed, the supply function becomes $p - t = c_1 + c_2 q$, where c_1 and c_2 are positive constants, and the total cost function becomes $c_t = c_f + (k+t)q$, where k is the cost of producing each unit. Here also $c_v = kq$.

Example 8. *Revenue* is the amount of money received when a firm sells its output. The relation is $r_t = pq$, where r_t is the total revenue, p the price and q the quantity.

Definition 7. *Elasticity* ε of x with regard to y means the ratio of the change in x to the change in y. Here y could be some economic variable, for example price or income. Hence, the *price elasticity of demand* is $\varepsilon_d = \frac{\mathrm{d}q_d}{\mathrm{d}p}\frac{p}{q_d}$, the *price elasticity of supply* $\varepsilon_s = \frac{\mathrm{d}q_s}{\mathrm{d}p}\frac{p}{q_s}$. The *point elasticity* is the elasticity value calculated at a point, the *arc elasticity* or the *midpoint elasticity* is the same averaged over an interval. The *arc price elasticity of demand* or *supply* is then $\varepsilon_d = \frac{\mathrm{d}q}{\mathrm{d}p}\frac{p_1+p_2}{q_1+q_2}$. The *arc income elasticity of demand* or *supply* is $\varepsilon_y = \frac{\mathrm{d}q}{\mathrm{d}y}\frac{y_1+y_2}{q_1+q_2}$.

§

Elasticity measures the sensitivity or responsiveness of a certain quantity to changes in some variable.

Definition 8. A *function of one independent variable* is a relation in the form $y = \mathrm{f}(x)$ such that there exists one and only one value of y in the range of f for each real number x in the domain of f. The variable y is called the *dependent variable.*

§

Definition 9. A function is called *multivalued function* if the opposite to Definition 8 is true, that is there exist more than one values of y for some x.

§

Definition 10. An *implicit function* is a function in which both dependent- and independent variables appear on the same side. An *explicit function* is one where the dependent variable is on the left hand side-, and the independent variable on the right hand side of the equation.

§

Example 9. Examples of explicit functions are $y = 5x$, $y = x^3 + x^2 - 7$ and $y = e^x + (x+1)\ln x$. Examples of implicit functions are $x + y = 1$, $x^2 + 3xy - y + y^2 = \frac{1}{x+y}$ and $x^2 \ln y = e^y(x + x^3)$.

Definition 11. The *exponential function* has the general form $y = a^x$, where the *base* a is a constant and x is called the *index-*, *power-*, or the *exponent* of the exponential function. The *logarithmic function* is then $\log_a y = x$, providing that both a and y are positive real numbers. We call a^x the x^{th} *power of* a, and call $\log_a y$ the *logarithm* of y to the base a. When the logarithmic base is the natural number, we write $\log_e n = \ln n$.

§

The logarithmic function is the inverse of the exponential function and vice versa.

Theorem 1. Let p and q be real numbers, a and b positive numbers, and m and n positive integers. Then,

$$a^p \cdot a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$(a^p)^q = a^{pq}$$

$$a^0 = 1, \text{ provided that } a \neq 0$$

$$a^{-p} = \frac{1}{a^p}$$

$$(ab)^p = a^p b^p$$

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

§

Theorem 2.

$$\log_a mn = \log_a m + \log_a n$$

$$\log_a \frac{m}{n} = \log_a m - \log_a n$$

$$\log_a m^p = p \log_a m$$

$$\log_a n = \frac{\log_b n}{\log_b a}$$

§

Both Definition 12 and Definition 13 are a definition of limits in calculus. Definition 13 is a more formal and definitive one.

Definition 12. If $\mathrm{f}(x)$ is a function which draws closer to a unique finite real number l for all values of x as the latter draws closer to a, but $x \neq a$, then l is called the *limit* of $\mathrm{f}(x)$ as x approaches a. In notation this is,

$$\lim_{n \to a} \mathrm{f}(x) = l$$

§

Definition 13. For a function $\mathrm{f}(x)$, $\lim_{n \to a} \mathrm{f}(x) = l$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathrm{f}(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

§

By Definition 13 we mean that one can get $\mathrm{f}(x)$ to be as close to l as one wish, provided that x gets close enough to a.

Theorem 3. Providing that both $\lim_{x \to a} \mathrm{f}(x)$ and $\lim_{x \to a} \mathrm{g}(x)$ exist, and let k be a constant and n a positive integer, then the *rules of limits* are the following.

$$\lim_{x \to a} k = k$$

$$\lim_{x \to a} x^n = a^n$$

$$\lim_{x \to a} k\mathrm{f}(x) = k \lim_{x \to a} \mathrm{f}(x)$$

$$\lim_{x \to a} [\mathrm{f}(x) \pm \mathrm{g}(x)] = \lim_{x \to a} \mathrm{f}(x) \pm \lim_{x \to a} \mathrm{g}(x)$$

$$\lim_{x \to a} [\mathrm{f}(x) \cdot \mathrm{g}(x)] = \lim_{x \to a} \mathrm{f}(x) \cdot \lim_{x \to a} \mathrm{g}(x)$$

$$\lim_{x \to a} \left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\lim_{x \to a} \mathrm{f}(x)}{\lim_{x \to a} \mathrm{g}(x)}, \text{ provided that } \lim_{x \to a} \mathrm{g}(x) \neq 0$$

$$\lim_{x\to a}[\mathrm{f}(x)]^n = \left[\lim_{x\to a}\mathrm{f}(x)\right]^n, \text{ provided that } n > 0$$

§

Definition 14. Let $y = \mathrm{f}(x)$. Then, the derivative of y with respect to x is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

The various notations for the derivative include $\mathrm{d}f(x)\mathrm{d}x$, $\frac{\mathrm{d}f}{\mathrm{d}x}$, $f'(x)$, y', $\mathrm{D}y$ and $\mathrm{D}(\mathrm{f}(x))$. The process for obtaining this is called *differentiation.*

§

Theorem 4. Let $y = x^n$. Then the *power rule* for differentiation is the following.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = nx^{n-1}$$

§

Theorem 5. The derivatives of exponential and logarithmic functions are respectively $\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$ and $\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$.

§

It turns out that e^x is the only function which is changed by neither differentiation nor integration. Leonhard Euler (1707–83) found that the *natural number* $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}^n\right) = 2.71828\ldots$.

Exercise 1. Find the first derivative of the following:

1. $y = 2e^x$
2. $y = 2\ln(x)$
3. $c_{vt} = \frac{100}{x} + \ln x$
4. $p = 100(1 - e^t)$
5. $q = \sqrt[4]{l} + 4$
6. $p = 1 + 1\ln q$
7. $q = \frac{\ln p}{8} - \frac{13}{\sqrt{4p}}$
8. $c_a = \ln q + \frac{1}{q}$
9. $p = 98.5\frac{e^{3t}}{e^t}$
10. $c = x^2 - e^x + \ln x$
11. $c = \frac{157}{y} + 0.81e^y$
12. $x = 34.3e^t$

Theorem 6. Let y be a function of u, and u a function of x. Then the *chain rule* states that,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

§

Exercise 2. Differentiate the following:

1. $p = 5 + \frac{8}{e^{4t}}$
2. $y = \frac{5}{2+3e^{4t}}$
3. $p = \ln\left(\frac{q^2-2q}{3q}\right)$
4. $y = (3x - 5)^7$
5. $q = 0.95\sqrt[3]{2}l^2 + 3l$
6. $c_t = \sqrt{q^5 + 4q^2}$
7. $y = (2 - 0.7x)^{-8}$
8. $t = 100 + 30e^{0.8y}$
9. $y = 11e^{-1.15} + \frac{1}{x+1} - \ln(3x^2 - 2x + 5)$
10. $p = \frac{3}{\sqrt{q}} + 2\ln(q)$
11. $s = 123(1 - 3e^{-0.62y})$
12. $y = \frac{1}{(5x+6)^{\frac{2}{3}}}$

Theorem 7. If $y = u(x)v(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

Similarly, if $y = u(x)v(x)w(x)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = uv\frac{\mathrm{d}w}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$$

§

Exercise 3. Find the first derivatives of the following:

1. $c = 3q\sqrt{2+q}$
2. $p = \frac{2}{33}x^2e^{x-7}$
3. $y = x^{1.3}\sqrt{4x}$
4. $y = (t^3)e^{2t} + 5$
5. $s = 10ye^{-0.1y}$
6. $c_a = \frac{1}{q} - q\ln q + \sqrt{q}$
7. $x = y^3\ln(y^3) + \sqrt{3y}\ln y$
8. $I = \frac{1}{y}\ln(3y+2)$
9. $p = q^2e^{q-3}\ln(q^2-8)$
10. $c_t = \frac{1}{q^3} + q\sqrt{q-1} - q + 100$
11. $y = t^2e^t + 10\ln t$
12. $s = \frac{e^y\ln y}{y}$

Theorem 8. If $y = \frac{u(x)}{v(x)}$, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

§

Exercise 4. Find the first derivative of the following and simplify your answer:

1. $y = \frac{t-1}{t+1}$
2. $t = \frac{1-t}{1+t}$
3. $t = \frac{\ln y}{y}$
4. $y = \frac{x\ln x+1}{x}$
5. $p = 100 - \frac{q}{q+1}$
6. $c = 500\left(1 - \frac{e^{-1.6y}}{y}\right)$
7. $p = \frac{q^3}{q+1}$
8. $c_t = \frac{q^2e^{6-q}}{(x^3+x^2+x+1)\ln q}$
9. $y = \frac{\sqrt{t-1}}{t}$

Definition 15. The *total cost* c_t comprises a fixed cost and a variable cost, that is $c_t = c_f + c_v$. The *total revenue* r_t is price times output, $r_t = pq$.

§

Definition 16. The *marginal cost* c_m is the change in total cost caused by the production of an additional unit. The *marginal revenue* r_m is the change in total revenue coming from the sale of an extra good. In other words, $c_m = \frac{\mathrm{d}c_t}{\mathrm{d}q}$ and $r_m = \frac{\mathrm{d}r_t}{\mathrm{d}q}$, where q is the output.

§

Definition 17. The *average cost* c_a is the total cost per unit output, $c_a = \frac{c_t}{q}$. The *average revenue* r_a is the total revenue per unit output, $r_a = \frac{r_t}{q}$.

§

Example 10. From Definition 15, $r_t = pq$, and from Definition 17, $r_t = r_a q$, therefore $p - r_a$. From Definition 17, $c_a = \frac{c_t}{q}$, and from Definition 15, $c_a = c_f + c_v$. Therefore $c_a = c_{af} + c_{av}$, where the average fixed cost $c_{af} = \frac{c_f}{q}$ and the average variable cost $c_{av} = \frac{c_v}{q}$.

Procedure 1 calculates and draws graphs of the total-, average-, and marginal costs, which are respectively c_t, c_a and c_m. Similar graphs of other quantities may be drawn likewise, for example ones for the total-, average-, and marginal products.

Procedure 1 *Total-, average-, and marginal graphs*

```
Given: $c_t(x)$
$c_a \leftarrow \frac{c_t}{x}$
$c_m \leftarrow c_t'$
for each $f \in \{c_t, c_a, c_m\}$ do
  find the critical values $\mathbf{x}_c$ for $f' = 0$
  find $f''$
  $n \leftarrow |\mathbf{x}_c|$
  for $i = 1$ to $n$ do
    if $f''(x_i^c) > 0$ then
      $f(x_i^c)$ is convex and is the relative minimum of $f$
    elseif $f''(x_i^c) < 0$ then
      $f(x_i^c)$ is concave and is the relative maximum of $f$
    endif
  endfor
  find inflection points $\mathbf{x}_f$ from $f'' = 0$
endfor
plot $c_t(x)$, then $c_a(x)$ and $c_m(x)$
```

§

Problem 1. Let the total cost function be $c = q^3 - 40q^2 + 800$. Sketch the graph to show the relationship between total-, average-, and marginal costs. If $c_t = \mathrm{f}(q)$, then $c_m = \mathrm{d}c_t/\mathrm{d}q$, and if $r_t = \mathrm{f}(q)$, then $r_m = \mathrm{d}r_t/\mathrm{d}q$, where c_t is the total cost, r_t the total revenue, and q the level of output.

Problem 2. The demand function for a monopolist is $q = 100 - 4p$. Find r_t, r_m and r_a, then evaluate these at $q = 10$ and explain the results. Further, find q when $r_a = 0$, then find r_m when $r_a = 0$, and then explain whether one should sell at this value of q. Draw graphs of r_t, r_m and r_a on the same diagram.

Problem 3. The demand function for a good is $p = 200 - q^{1.8}$. Find r_t, r_m and r_a, and then compare the slopes of the r_m- and r_a curves. Evaluate r_t, r_m and r_a at $q = 11$ and at $q = 26$, and give an explanation of the values obtained. Find q when $r_m = 0$, and find q when $r_a = 0$. When does the sales of more units begin to reduce the total revenue? Draw graphs of r_t, r_m and r_a on the same diagram.

Problem 4. The fixed costs of a firm are 2000 and the variable costs are $4.7q$. Write the equation for c_t and evaluate the c_t at $q = 35$. Write the equation for c_m and evaluate the c_m at $q = 35$. Explain the meaning of c_m for this function.

Problem 5. The average cost function of a firm is given by $c_a = q^2 - 10q + \frac{210}{q} + 66$. Write expressions for c_t and c_m, and then calculate c_t at $q = 17$. Write the equations for c_f and c_{vt}.

Problem 6. The average cost function of a firm is $c_a = 97 + \frac{21}{q}$. Differentiate c_a with respect to q, then describe and explain how the former changes with the change in the latter.

Definition 18. The relationship between input and output is called a *production function*, $q = \mathrm{f}(l, k, r, t_e, s, e, \ldots)$, where l is labour, k phical capital such as buildings and machines, r raw materials, t_e technology, s land, and e enterprise. Assuming a short period of time, then l becomes the only independent variable and the other remaining factors are parameters, that is fixed, and therefore $q = \mathrm{f}(l)$. Then the *marginal product of labour* is $p_{lm} = \frac{\mathrm{d}q}{\mathrm{d}l}$, and the *average product of labour* is $p_{la} = \frac{q}{l}$.

§

The value p_{la} in Definition 18 is a measure of productivity.

Definition 19. The *marginal propensity to consume* is $p_{cm} = \frac{\mathrm{d}c}{\mathrm{d}y}$. The *marginal propensity to save* is $p_{sm} = \frac{\mathrm{d}s}{\mathrm{d}y}$. The *average propensity to consume* is $p_{ca} = \frac{c}{y}$. The *average propensity to save* is $p_{sa} = \frac{s}{y}$. Here y is the income, c the consumption, and s the saving.

§

Example 11. Since $y = c + s$, it follows that $p_{ca} + p_{sa} = 1$. Further, $\frac{\mathrm{d}y}{\mathrm{d}y} = \frac{\mathrm{d}c}{\mathrm{d}y} + \frac{\mathrm{d}s}{\mathrm{d}y}$, and hence $1 = p_{cm} + p_{sm}$.

Example 12. The consumption function may be described by $c = c_0 + by$, where c_0 and b are positive constants. Then $p_{ca} > p_{cm}$ and $p_{sm} > p_{sa}$.

Definition 20. The *profit* is $\pi = r_t - c_t$. At the *break-even point* $\pi = 0$, that is $r_t = c_t$.

§

Problem 7. Find the elasticity of demand when $q = \frac{a}{p^c}$, where a and c are constants.

Problem 8. The demand function for train journeys is $q = \frac{1600}{p^{1.3}}$, where q is the number of passengers in thousands. Find the elasticity of demand in this case. Determine the percentage change in demand when the fare increases or decreases by 10 per cent.

Problem 9. The demand for mineral water is given by $q = 100 - 2p$, where p is the price per bottle and q the number of bottles demanded. Find the expressions for r_t, r_m and r_a. Find the quantity and price that maximise the revenue. Find an equation for the price elasticity of demand firstly in terms of p, and then again in terms of q. Show that r_t is maximum and r_m is zero when $\varepsilon_d = -1$, and find q at this ε_d.

Problem 10. A demand for a wine is given by $p = 1200e^{-0.03q}$, where p is the price per bottle and q the number of bottles demanded. Write the equations for r_t, r_m and r_a. Find the price and quantity when the revenue is maximised. Show that $\varepsilon_d = -1$ when r_t is maximised.

Problem 11. Derive the equation

$$r_m = p\left(1 + \frac{1}{\varepsilon_d}\right)$$

Problem 12. Show that the profit becomes maximised when

$$p = \frac{c_m}{\left(1 + \frac{1}{\varepsilon_d}\right)}$$

Problem 13. A shop selling shirts has a demand function $p = 300 - 10q$ and a total cost function $c_t = 150 + 8q$. Find the equations for r_t and π. Find the number of shirts which must be sold firstly in order to maximise the profit, and then again to maximise the total revenue. Show that $r_m = r_c$ when the profit is maximised.

Problem 14. The average cost function of a mobile phone is $c_a = 10 + \frac{5000}{q}$, and the average revenue function for the same is $r_a = 35$. Find r_t, c_t, r_m and c_m. How many mobile phones must be made and sold in order to break-even? Find the profit function, and show that neither profit nor revenue has a maximum. Explain using r_m and c_m why there is no maximum. Plot the graphs for r_t, c_t and π on one diagram, and plot r_m and c_m on another diagram. Comment on the graphs.

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Calculus of multivariable functions

1^{st} November 2005

Definition 21 talks about functions of n independent variables, and Definition 22 the partial derivatives of these. In Theorem 9 we have the product rule, in Theorem 10 the quotient rule, and in Theorem 11 the generalised power function rule.

Definition 21. A function $y = \mathrm{f}(x_1, \ldots, x_n)$ is called a *function of* n *independent variables* if there exists one and only one value of y in the range of f for each tuple of real number $(x_1, \ldots, x_n)$ in the domain of f. Here y is called the *dependent variable* while x_i, $i = 1, \ldots, n$, the *independent variables.*

§

The word *tuple* in Definition 21 means an *ordered list.* It is also known as n-tuple, where n is the size of the list.

Definition 22. Let a multivariable function be $y = \mathrm{f}(x_1, \ldots, x_n)$. The *partial derivative* of y with respect to x_i, where $1 \leq i \leq n$, is a measure of the instantaneous rate of change of y with respect to x_i while x_j is held constant for all $j \neq i$, where $1 \leq j \leq n$. This partial derivative is defined as

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\mathrm{f}(\ldots, x_i + \Delta x_i, \ldots) - \mathrm{f}(x_1, \ldots, x_n)}{\Delta x_i}$$

and can be written in either one of the following forms.

$$\frac{\partial y}{\partial x_i}, \ \frac{\partial \mathrm{f}}{\partial x_i}, \ \mathrm{f}_{x_i}(x_1, \ldots, x_n), \ \mathrm{f}_{x_i}, \text{ or } y_{x_i}$$

§

Theorem 9. Let $z = \mathrm{g}(x, y) \cdot \mathrm{h}(x, y)$. Then,

$$\frac{\partial z}{\partial x} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial x} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial y} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

§

Theorem 10. Let $z = \frac{\mathrm{g}(x,y)}{\mathrm{h}(x,y)}$ and $\mathrm{h}(x, y) \neq 0$. Then,

$$\frac{\partial z}{\partial x} = \frac{\mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial x} - \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial x}}{\mathrm{h}^2}$$

and

$$\frac{\partial z}{\partial y} = \frac{\mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial y} - \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial y}}{\mathrm{h}^2}$$

§

Theorem 11. Let $z = [\mathrm{g}(x,y)]^n$. Then,

$$\frac{\partial z}{\partial x} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

§

Exercise 5. Find the first-order partial derivatives of the following:

1. $c = 100(1 + e^{-1.2q})$
2. $q = 5l^{0.7}k^{0.3} - 4l - 3k + 10$
3. $q = \ln 3x + 2x \ln y$
4. $z = x^3(1 + y + y^2)$
5. $p = 150e^{0.74t}$
6. $q = lnx + \ln y$
7. $z = x^3 + x^2 + x + 2xy + xy^2$
8. $z = x^2y^5$
9. $q = 2l^{0.77}k^{0.1}$
10. $u = 8x^3y^3$
11. $z = xy + \frac{y}{x}$
12. $u = 7l^7k^4$

§

In Definition 23 we find the meaning of second-order partial derivatives. Theorem 12 is about critical points, and Procedure 2 is a procedure for determining critical points.

Definition 23. Let $z = \mathrm{f}(x,y)$. Then, the *second-order direct partial derivatives* are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

These are also written

$$\mathrm{f}_{xx},\ (\mathrm{f}_x)_x,\ \frac{\partial^2 z}{\partial x^2} \quad \text{and respectively} \quad \mathrm{f}_{yy},\ (\mathrm{f}_y)_y,\ \frac{\partial^2 z}{\partial y^2}$$

The *cross partial derivatives* are

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

These are also written as

$$\mathrm{f}_{xy},\ (\mathrm{f}_x)_y,\ \frac{\partial^2 z}{\partial y \partial x} \quad \text{and respectively} \quad \mathrm{f}_{yx},\ (\mathrm{f}_y)_x,\ \frac{\partial^2 z}{\partial x \partial y}$$

§

Exercise 6. Find the second-order partial derivatives of the following:

1. $z = x^3 + xy$
2. $z = l^{0.2}k^{0.3} - \lambda(10 - l - 3k)$
3. $z = 2x^3y^4$
4. $c = 100(1 + e^{-0.1q})$
5. $q = \ln x + \ln y$
6. $q = 5l^{0.44}k^{0.2} + 50 - 7l - 5k$
7. $q = 9l^{0.77}k^{0.1}$
8. $q = \ln 5x + x \ln y$
9. $p = 110e^{0.87t}$
10. $z = xy + \frac{y}{2x}$
11. $u = 10x^4y^4$
12. $z = x^3(1 + y + y^3)$

§

When the second derivative is negative, the curve is concave towards the origin.

Theorem 12. For a multivariable function $z = \mathrm{f}(x, y)$ to be a *relative maximum* at (a, b) necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} < 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ at that point. For the same at the same to be a *relative minimum*, necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} > 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ there. Moreover, an *inflection point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, and both f_{xx} and f_{yy} have the same sign. On the other hand, a *saddle point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, but f_{xx} and f_{yy} are of different signs.

§

Procedure 2 *Procedure for determining a critical point of a function with two independent variables*

```
Given $z = \mathrm{f}(x, y)$ and a point $(a, b)$, at this point,
if $\mathrm{f}_x = 0$ and $\mathrm{f}_y = 0$ then
  $(a, b)$ is a critical point
  if $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ then
    if $\mathrm{f}_{xx} < 0$ and $\mathrm{f}_{yy} < 0$ then
      $(a, b)$ is a relative maximum of $z$
    elseif $\mathrm{f}_{xx} > 0$ and $\mathrm{f}_{yy} > 0$ then
      $(a, b)$ is a relative minimum of $z$
    else †
    endif
  elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$ then
    if $\mathrm{f}_{xx} \cdot \mathrm{f}yy > 0$ then
      $(a, b)$ is an inflection point
    elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < 0$ then
      $(a, b)$ is a saddle point
    else ‡
    endif
  else
    test inconclusive
  endif
else
  $(a, b)$ is no critical point
endif
```

§

Problem 15. There are two dead ends in Procedure 2. The first one (†) is the case where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ and either $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} < 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} > 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} > 0)$, or $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} < 0)$. The second one (‡) is where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} = 0$. Find out what happen in these cases, and thus complete the missing lines of logic in Procedure 2.

§

Definition 24 gives the meaning of derivative and differential. Example 13 looks at the derivative & differential of functions of one variable & two variables.

Definition 24. By *derivative* $\frac{\mathrm{d}y}{\mathrm{d}x}$ we mean an infinitesimally small change in y with respect to an infinitesimally small change in x. By *differential* $\mathrm{d}y$ and $\mathrm{d}x$ we mean an infinitesimally small change in the values of y and respectively x.

§

Example 13. For a function of one variable $y = \mathrm{f}(x)$, the total derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

and the differential of y is

$$\mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

For a function of two variables $z = \mathrm{f}(x, y)$ partial derivatives are, the first-order partial derivatives

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}$$

and the second-order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx}, \ \frac{\partial^2 z}{\partial y^2} \equiv z_{yy}, \ \frac{\partial^2}{\partial y \partial x} \equiv z_{xy} \text{ and } \frac{\partial^2 z}{\partial x \partial y} \equiv z_{yx}$$

The total differential of z is

$$\mathrm{d}z = \left(\frac{\partial \mathrm{f}}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial \mathrm{f}}{\partial y}\right)\mathrm{d}y$$

and for small changes which are not infinitesimal, $\mathrm{d}x$ becomes Δx and the incremental change formula is

$$\Delta z \approx \left(\frac{\partial \mathrm{f}}{\partial x}\right)\Delta x + \left(\frac{\partial \mathrm{f}}{\partial y}\right)\Delta y$$

Definition 25 gives the general production function. In Example 14 we look at the Cobb-Douglas production function in more details. Theorem 13 gives the law of diminishing returns to labour and the proof thereof, while similarly does Theorem 14 the law of diminishing returns to capital.

Definition 25. The *general production function* is $q = \mathrm{f}(l, k)$, where q is output of the production, l labour and k capital. The *Cobb-Douglas production function* in its general form is

$$q = al^{\alpha}k^{\beta} \tag{1}$$

where a is a constant and $0 < \alpha < 1$, $0 < \beta < 1$, $l > 0$ and $k > 0$.

§

Example 14. With the Cobb-Douglas production function, the *marginal product of labour* is,

$$p_{lm} = q_l = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{2}$$

and the *marginal product of capital*

$$p_{km} = q_k = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{3}$$

From this we see that $p_{lm} > 0$ and $p_{km} > 0$.

Theorem 13. From the Cobb-Douglas production function we have the *law of diminishing returns to labour*, which states that $q_{ll} < 0$.

Proof. From Equation 1 in Definition 25,

$$q_{ll} = \frac{\partial^2 q}{\partial l^2} = \frac{\partial}{\partial l}\left(\frac{\partial q}{\partial l}\right) = \frac{\partial p_{lm}}{\partial l} = (\alpha - 1)\frac{\alpha q}{l^2}$$

Since $0 < \alpha < 1$, therefore $q_{ll} < 0$. ¶

Theorem 14. Using the Cobb-Douglas production function, the *law of diminishing returns to capital* states that $q_{kk} < 0$.

Proof. From Equation 1 in Definition 25,

$$q_{kk} = \frac{\partial^2 q}{\partial k^2} = \frac{\partial}{\partial k}\left(\frac{\partial q}{\partial k}\right) = \frac{\partial p_{km}}{\partial k} = (\beta - 1)\frac{\beta q}{k^2}$$

which, with $0 < \beta < 1$, tells us that $q_{kk} < 0$. ¶

In Example 15 we see the changes in marginal product values.

Example 15. Using the Cobb-Douglas production function,

$$q_{kl} = q_{lk} = a\alpha\beta l^{\alpha-1}k^{\beta-1}$$

Therefore, $q_{lk} > 0$ and $q_{kl} > 0$. In other words, p_{lm} increases as capital input k increases, and respectively p_{km} increases as labour input l increases.

Example 16 shows us the average functions of labour and capital, Example 17 the marginal functions of labour and capital, and Example 18 the comparison between marginal and average functions.

Example 16. For the Cobb-Douglas production function in Equation 1 the average product of labour is

$$p_{la} = \frac{q}{l} = al^{\alpha-1}k^{\beta} \tag{4}$$

and the average product of capital is

$$p_{ka} = \frac{q}{k} = al^{\alpha}k^{\beta-1} \tag{5}$$

Example 17. Again using the Cobb-Douglas production function of Equation 1, the marginal product of labour is

$$p_{lm} = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{6}$$

and the marginal product of capital is

$$p_{km} = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{7}$$

Example 18. From the APL equation, Equation 4, and the MPL equation, Equation 6, and since $0 < \alpha < 1$, therefore $p_{ml} < p_{la}$. Similarly from the APK equation, Equation 5, and the MPK equation, Equation 7, since $0 < \beta < 1$, we have $p_{km} < p_{ka}$.

In Example 19 one sees the conditions for using labour, Equation 8, and the conditions for using capital, Equation 9.

Example 19. A producer likes to have a positive marginal function, which means that the productivity increases as the input increases. But the second derivative is negative, which means that this rate of increase slows down as time goes by. In practice, the conditions for using labour are,

$$p_{lm} = \frac{\partial q}{\partial l} > 0,\ \frac{\mathrm{d}p_{lm}}{\mathrm{d}l} = \frac{\partial^2 q}{\partial l^2} < 0,\ \text{and}\, p_{lm} < p_{la} \tag{8}$$

The conditions for using capital are similarly,

$$p_{km} = \frac{\partial q}{\partial k} > 0,\ \frac{\mathrm{d}p_{km}}{\mathrm{d}k} = \frac{\partial^2 q}{\partial k^2} < 0,\ \text{and}\, p_{km} < p_{ka} \tag{9}$$

Definition 26 and Theorem 15 deal respectively with production function graphs and slope of an isoquant.

Definition 26. An *isoquant* is a graph in two dimensions, $k = \mathrm{f}(l)$, plotted to represent a production function $q = \mathrm{f}(l, k)$. The slope $\frac{\mathrm{d}k}{\mathrm{d}l}$ is called the *marginal rate of technical substitution*. The value of this slope at (l_0, k_0) is denoted by $\frac{\mathrm{d}k}{\mathrm{d}l}\big|_{l_0 k_0}$.

§

Theorem 15. The slope of an isoquant is the ratio of the marginal products.

Proof. The total differential of $q = \mathrm{f}(l, k)$ is

$$\mathrm{d}q = \left(\frac{\partial q}{\partial l}\right) \mathrm{d}l + \left(\frac{\partial q}{\partial k}\right) \mathrm{d}k$$

Along any isoquant, $\mathrm{d}q = 0$, therefore,

$$0 = \left(\frac{\partial q}{\partial l}\right) \mathrm{d}l + \left(\frac{\partial q}{\partial k}\right) \mathrm{d}k \tag{10}$$

This directly yield, after some manipulation,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{q_l}{q_k}$$

Or, from Equation 10 together with Equation's 6 and 7, it follows that,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{p_{lm}}{p_{km}}$$

Definition 27. In the Cobb-Douglas production function equation, Equation 1, let both inputs l and k change by the same proportion, and let λ be the constant of this proportionality. Then $q_2 = a(\lambda l)^{\alpha}(\lambda k)^{\beta}$, which leads to $q_2 = \lambda^{\alpha+\beta} q_1$. When $\alpha + \beta = 1$, the case is described as *constant returns to scale*, when $\alpha + \beta < 1$ as *decreasing returns to scale*, and when $\alpha + \beta > 1$ as *increasing returns to scale.*

§

Definition 28. A *homogeneous* Cobb-Douglas production function of order r is,

$$\mathrm{f}(\lambda l, \lambda k) = \lambda^r \mathrm{f}(l, k)$$

where $r = (\alpha, \beta)$.

§

The utility function in its general form is given in Definition 29. Definition 30 gives the Cobb-Douglas utility function, Definition 31 the marginal utility, and Definition 32 the meaning of indifference curves.

Definition 29. A *utility function* expresses utility as a function of goods consumed. In its general form this is,

$$u = \mathrm{f}(x, y)$$

where x and y are the quantities of goods X and respectively Y consumed.

§

Definition 30. The *Cobb-Douglas utility function* is in its general form,

$$u = ax^{\alpha}y^{\beta}$$

where a is a constant, and $0 < \alpha < 1$, $0 < \beta < 1$, $x > 0$ and $y > 0$.

§

Definition 31. The *marginal utility* for a utility function with one variable, $u = \mathrm{f}(x)$, is $\frac{\mathrm{d}u}{\mathrm{d}x} = u_x = u_{xm}$. The marginal utility for a utility function with two variables, $u = \mathrm{f}(x, y)$, is $\frac{\partial u}{\partial x} = u_x = u_{xm}$ and $\frac{\partial u}{\partial y} = u_y = u_{ym}$.

§

Definition 32. The *indifference curve* is a graph $y = \mathrm{f}(x)$ drawn to represent a utility function $u = \mathrm{f}(x, y)$. Its slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the *marginal rate of substitution.* Setting the total differential equal to zero,

$$0 = \mathrm{d}u = \left(\frac{\partial u}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial u}{\partial y}\right)\mathrm{d}y$$

we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_x}{u_y}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_{xm}}{u_{ym}}$$

§

Partial elasticities are describe in Definition 33, partial elasticities of demand, and Example's 20 and 21, respectively the partial elasticity with respect to labour and the partial elasticity with respect to capital.

Definition 33. Let a demand function be

$$q_a = \mathrm{f}(p_a, y, p_b) \tag{11}$$

where q_a is the quantity demanded of good a, p_a the price of a, y consumer's income, and p_b the price of another good b. Then, the *price elasticity of demand* is,

$$\varepsilon_d = \frac{\partial q_a}{\partial p_a}\frac{p_a}{q_a}$$

The *income elasticity of demand* is,

$$\varepsilon_y = \frac{\partial q_a}{\partial y}\frac{y}{q_a}$$

And the *cross-price elasticity of demand* is,

$$\varepsilon_c = \frac{\partial q_a}{\partial p_b}\frac{p_b}{q_a}$$

§

Example 20. With the demand function as in Equation 11, the partial elasticity with respect to labour is,

$$\varepsilon_{ql} = \frac{\partial q}{\partial l}\frac{l}{q}$$

And from Equation's 6 and 4, this leads to,

$$\varepsilon_{ql} = \frac{p_{lm}}{p_{la}}$$

For the Cobb-Douglas production function, Equation 1, then $\varepsilon_{ql} = \alpha$.

§

Example 21. Again, with the demand function as in Equation 11, the partial elasticity with respect to capital is,

$$\varepsilon_{qk} = \frac{\partial q}{\partial k}\frac{k}{q}$$

Then, from Equation's 7 and 5,

$$\varepsilon_{qk} = \frac{p_{km}}{p_{ka}}$$

For the Cobb-Douglas production function, Equation 1, we have $\varepsilon_{qk} = \beta$.

§

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Exponential, logarithmic and nonlinear functions

8th November 2005

Definition 34 gives a definition of a function. Example 22 gives some examples of this. In Definition 35 we talk about variables and parameters of a function.

Definition 34. A *function* is an operator or a procedure which accepts a permissible input and transforms it into a unique output. The input is some nonempty set. If a function is defined to be $y = \mathrm{f}(\mathrm{x})$, then x is the input vector, y the output, and $\mathrm{f}(\cdot)$ the function itself.

§

Example 22. If $\mathrm{f}(\cdot)$ is the function of dressing, then its input is possibly a person and its output a dressed person. If $\mathrm{f}(\cdot)$ is the function of making up, then the input is perhaps a girl and the output a made-up girl.

Definition 35. Let $y = \mathrm{f}(\mathrm{a}, \mathrm{x})$ be a function, where a is a set of all its parameters, and x a set of all its variables. Then y is its dependent variable and x_i, for all $i \in \mathrm{x}$, are its independent variables. In other words, x_i vary, y follows, and a_i could assume any value within the range of its permissible ones, but its value must be constant.

§

Definition 36 and Example 23 address inverse function, and respectively operator and inverse operator.

Definition 36. An *inverse function* is an expression of the independent variable in terms of the dependent variables. The inverse of the function $\mathrm{f}(\cdot)$ is denoted by $\mathrm{f}^{-1}(\cdot)$. If $\mathrm{f}(\cdot)$ is a function which admits one independent variable, namely x, then one could express it as,

$$y = \mathrm{f}(x) \tag{12}$$

Its inverse function is then,

$$\mathrm{f}^{-1}(y) = x \tag{13}$$

§

Example 23. Both the function and its inverse may be thought of as being an operator operating on an input to produce an output. The function,

$$y = \mathrm{f}(x)$$

is understood diagrammatically as,

$$y \leftarrow \boxed{\mathrm{f}(\cdot)} \leftarrow x$$

while its inverse function,

$$x = \mathrm{f}^{-1}(y)$$

is displayed as a diagram as,

$$y \rightarrow \boxed{\mathrm{f}^{-1}(\cdot)} \rightarrow x$$

Theorem 16 is related to the domain and range of inverse functions. Example 24 gives some examples of inverse functions.

Theorem 16. An inverse function must always be a one-to-one mapping.

Proof. Let $\mathrm{f}(\cdot)$ be a function. Then $\mathrm{f}(\cdot)$ can be either one-to-one or many-to-one, and therefore $\mathrm{f}^{-1}(\cdot)$ could turn out to be either one-to-one or one-to-many. But since f^{-1} is also a function, so for each of the values in its domain the corresponding value in its range must be unique. This means that in cases where f^{-1} turns out to be one-to-many, some constraints must be put on its input in order to make the output one-to-one, which then makes all the outputs from $\mathrm{f}^{-1}(\cdot)$ one-to-one. ¶

Example 24. Table 1 gives some of the functions and their corresponding inverse functions which are fundamental in mathematics.

$\mathrm{f}(\cdot)$	$\mathrm{f}^{-1}(\cdot)$
addition	subtraction
multiplication	division
power	root
exponential	logarithm

Table 1 *Some of the functions and their corresponding inverse functions.*

Then, letting a be a constant, Table 1 becomes Table 2

$\mathrm{f}(\cdot)$	$\mathrm{f}^{-1}(\cdot)$
$x + a$	$x - a$
$x \cdot a$	$\frac{x}{a}$
x^a	$\sqrt[a]{x}$
a^x	$\log_a x$

Table 2 *The notational forms of functions and their inverses.*

in which division and logarithm are both undefined for $a = 0$.

Definition 37 gives some of the basic building blocks of mathematics. Example 25 then shows how these are built one on top of another.

Definition 37. The inverse of the addition,

$$y = x + a$$

is the subtraction,

$$y - a = x$$

The inverse of the multiplication,

$$y = ax$$

is the division,

$$\frac{y}{a} = x$$

The inverse of the power,

$$y = x^a$$

is the root,

$$\sqrt[a]{y} = x$$

The inverse of the exponential,

$$y = a^x$$

is the logarithm,

$$\log_a y = x$$

§

Example 25. Figure 10 is a diagram which shows how addition makes multiplication makes power function.

$$\begin{tikzcd}
 & \underbrace{x \cdots x}_{b} \arrow[r] & x^b \\
 & x \cdot x = x^2 \arrow[u] & \\
\underbrace{x + \ldots + x}_{a} \arrow[r] & x \cdot a \arrow[u] & \\
x + x = 2x \arrow[u] & &
\end{tikzcd}$$

Figure 10 *The building blocks of mathematics, from addition to multiplication to power function.*

In Figure 10 we start from considering a variable at the base. If instead of doing this we begin by considering addition of some constant a, then eventually a becomes a parameter in our more complicated functions.

$$\begin{tikzcd}
 & \underbrace{a \cdots a}_{x} \arrow[r] & a^x \\
\underbrace{a + \ldots + a}_{n} \arrow[r] & a \cdot n \arrow[u] & \\
a + a = 2a \arrow[u] & &
\end{tikzcd}$$

Figure 11 *Starting from a constant to obtain in the end the exponential function.*

Our derivation in Figure 10 gives us x^b when b is an integer, and similarly that in Figure 11 gives a^x when x is an integer, but both the power- and the exponential functions can be extended to cover cases where the powers are noninteger, that is to say, when they are real or complex numbers. In these cases, however, the output may no longer be real.

Next, we look at the exponential function, which is defined in Definition 38. Example 26 discusses this further, and the exponential to the power of zero is looked at in Theorem 17.

Definition 38. An *exponential function* is defined as $y = a^x$, where $a > 0$ and $a \neq 1$.

§

Example 26. The domain of the exponential function $y = a^x$ is the set of all real numbers, while its range the set of all positive real numbers. The function is convex and increasing when $a > 1$, and convex and decreasing when $0 < a < 1$. At $x = 0$, the value of the function is $y = 1$ for any $a > 0$.

Theorem 17. For any $a \neq 0$,

$$\lim_{x \to 0} a^x = 1$$

§

Problem 16. Try prove Theorem 17.

§

Theorem 18 gives some of the rules of exponential function and Example 26 looks at some exponential function.

Theorem 18. Three basic rules of the exponential function are,

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

Proof. Write, say, a^m as,

$$\underbrace{a \cdots a}_{m}$$

and similarly for a^n. Then all three equations above become obvious. ¶

Example 27. Figure 12 gives a graph of the exponential function when $a > 1$. Figure 13 gives a graph of the exponential function $y = a^x$ when $0 < a < 1$.

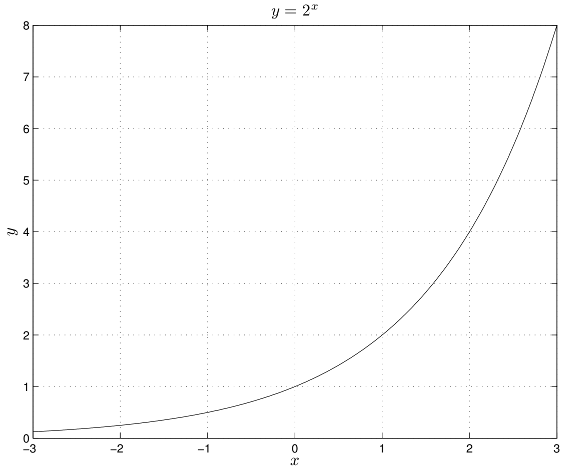

Figure 12 *Example of the graph of the exponential function when* $a > 1$*. Here the graph is that of* $y = 2^x$*.*

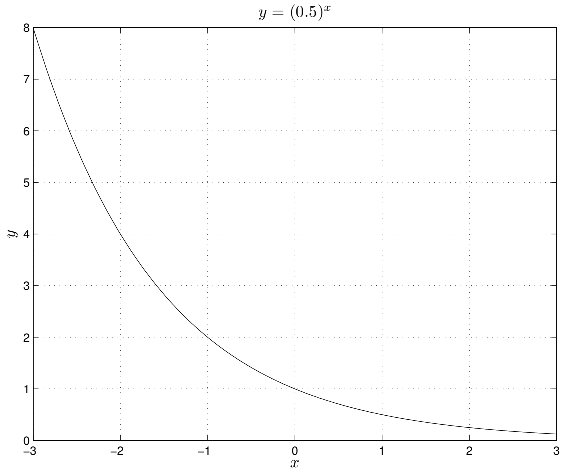

Figure 13 *An example of graph of the exponential function* $y = a^x$ *when* $0 < a < 1$*. Here* $a = 0.5$*.*

From Figure's 12 and 13, one may see that the graph of $y = a^x$, where $0 < a < 1$, is the same as the graph of $y = b^{-x}$, where $\frac{1}{a} = b > 1$. This is obvious since by putting $a = \frac{1}{b}$ into $y = a^x$ one arrives at $y = b^{-x}$, and if $0 < a < 1$, then $b > 1$.

One of the place we find use of an exponential function is the growth- and decay curves mentioned in Definition 39. Example 28 gives several basic growth functions.

Definition 39. Let $a > 1$. Then the graph of $y = a^x$ is called a *growth curve*, while that of $y = a^{-x}$ is called a *decay curve*.

§

Example 28. There are basically three laws of growth, namely unlimited, limited and logistic growth, all of which involve an exponential function. The model is for unlimited growth,

$$y(t) = ae^{rt}$$

for limited growth,

$$y(t) = m\left(1 - e^{-rt}\right)$$

and for logistic growth,

$$y(t) = \frac{m}{1 + ae^{-rmt}}$$

where a, m and r are constants.

Interests, and future- and present values are discussed in Example's 29 and 30.

Example 29. The value of a principal p compounded annually at an interest rate i for t years is,

$$s = p(1+i)^t$$

where i is expressed in decimal points. For compounding m times a year, then,

$$s = p\left(1 + \frac{i}{m}\right)^{mt}$$

If the compounding is continuous, at 100 per cent interest for one year, then,

$$s = p\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m = pe$$

where e is the Euler's constant, $e = 2.71828\ldots$

Example 30. For multiple compounding,

$$p(1+i_e)^t = p\left(1 + \frac{i}{m}\right)^{mt}$$

the effective annual rate of interest is,

$$i_e = \left(1 + \frac{i}{m}\right)^m - 1$$

The effective annual rate of interest for continuous compounding is,

$$i_e = e^r - 1$$

Definition 40 and Example 31 are about discounting.

Definition 40. *Discounting* is the process of finding the present value p of a future sum of money s.

§

Example 31. Discounting when under annual compounding is,

$$s = p(1+i)^t$$

when under multiple compounding,

$$p = s\left(1+\frac{i}{m}\right)^{-mt}$$

and when under continuous compounding,

$$p = se^{-rt}$$

When discounting, the interest rate i is called the *rate of discount.*

Example 32. A discrete growth $s = p(1+i/m)^{mt}$ can be converted to a continuous growth $s = pe^{rt}$ thus,

$$\begin{aligned} p\left(1+\frac{i}{m}\right)^{mt} &= pe^{rt} \\ \ln\left(1+\frac{i}{m}\right)^{mt} &= \ln e^{rt} \\ r &= m\ln\left(1+\frac{i}{m}\right) \end{aligned}$$

Therefore,

$$s = p\left(1+\frac{i}{m}\right)^{mt} = pe^{m\ln\left(1+\frac{i}{m}\right)t}$$

Example 33. Reversing the sign of x, that is replacing x by $-x$, has the effect of reflection of the original graph with respect to the y-axis. Reversing the sign of y, that is replacing y by $-y$, gives a reflection of the same with respect to the x-axis. The graphs of $y = a^{\pm x}$ remain always above the x-axis, in other words the function $y = a^{\pm x}$ maps $-\infty < x < \infty$ to $y > 0$. The two functions $y = a^x$ and $y = a^{-x}$ are the reflection of each other with respect to the y-axis. It can be easily seen that the functions $y = -a^{\pm x}$ are the reflection with respect to the x-axis respectively of $y = a^{\pm x}$.

Definition 41 introduces the logarithmic function, and Example 34 gives some elaboration regarding this. Some examples of natural logarithm are given in Example 35. Theorem 19 gives rules for logarithm.

Definition 41. The *logarithmic function* with base a is defined to be the inverse of the exponential function, and is written $y = \log_a x$, where $a > 0$ and $a \neq 1$. The logarithmic function of base 10 is called the *common logarithmic function*, and one of base e, where $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ is called the *natural logarithmic function.* By the notation $y = \log_a x$ we mean that the logarithm base a of x is the power to which a must be raised to get x.

§

Example 34. The domain of the logarithmic function $y = \log_a x$ is the set of all positive real numbers, its range the set of all real numbers. The function is concave and increasing for $a > 1$, and is convex and decreasing for $0 < a < 1$. Note also that $\log_a x$ is the power which a must be raised to get x.

Example 35. Note that $e^{\ln a} = a = \ln e^a$ where $a > 0$, $e^{\ln x} = x = \ln e^x$ where $x > 0$, and $e^{\ln \mathrm{f}(x)} = \mathrm{f}(x) = \ln e^{\mathrm{f}(x)}$ where $\mathrm{f}(x) > 0$.

Theorem 19. Four basic rules for logarithm function are listed in the following.

$$\log_b m + \log_b n = \log_b mn$$

$$\log_b m - \log_b n = \log_b \frac{m}{n}$$

$$\log_b m^z = z \log_b m$$

$$\log_b n = \frac{\log_x n}{\log_x b}$$

§

Problem 17. Prove Theorem 19, the theorem for rules of logarithm.

§

Definition 42 sets out the meaning of the elasticity of substitution. Example 36 discuss the values of the elasticity of substitution. Definition 43 is on the constant elasticity of substitution production function.

Definition 42. The *elasticity of substitution* σ is defined as,

$$\sigma = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\frac{k}{l}}}{\frac{\mathrm{d}\left(\frac{p_l}{p_k}\right)}{\frac{p_l}{p_k}}} = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\mathrm{d}\left(\frac{p_l}{p_k}\right)}}{\frac{\frac{k}{l}}{\frac{p_l}{p_k}}}$$

where $\frac{k}{l}$ is called the *least-cost input ratio*, and $\frac{p_l}{p_k}$ the *input-price ratio.*

§

Example 36. The value $\sigma = 0$ means there is no substitutability, that is the two inputs are complements of each other and both must be used together in a fixed proportion. The value $\sigma = \infty$ means that the two goods may substitute each other perfectly. Ultimately, $0 \leq \sigma \leq \infty$.

Definition 43. A *constant elasticity of substitution production function* is a production function where, unlike the Cobb-Douglas function, has an elasticity of substitution whose value is constant but not necessarily 1. In its typical form, it is,

$$q = a\left(\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right)^{-\frac{1}{\beta}}$$

where a is called the *efficiency parameter*, α the *distribution parameter*, β the *substitution parameter*. Furthermore, β determines σ, and $a > 0$, $0 < \alpha < 1$, and $\beta > -1$.

§

Example 37 discusses logarithmic transformation of nonlinear functions.

Example 37. Some nonlinear functions can be converted to linear functions using logarithmic transformation, for example the Cobb-Douglas production function,

$$q = ak^{\alpha}l^{\beta}$$

which becomes

$$\ln q = \ln a + \alpha \ln k + \beta \ln l$$

Other nonlinear functions can not be converted, for example the constant elasticity of substitution production function,

$$q = a\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]^{-\frac{1}{\beta}}$$

which becomes just another nonlinear function,

$$\ln q = \ln a - \frac{1}{\beta}\ln\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]$$

others

Example's 38 and 39 give some examples of the use of nonlinear function in economics, namely the nonlinear total revenue and the nonlinear total cost.

Example 38. Let the total revenue be $r_t = pq$, and the demand function $p = a - bq$, where q is the quantity sold. Then r_t expressed as a function of q is nonlinear, for $r_t = (a-bq)q = aq - bq^2$.

Example 39. A more realistic equation for the total cost instead of $c_t = a + bq$ is the nonlinear function $c_t = aq^3 - bq^2 + cq + d$ in which the production cost increases with quantity in at a decreasing rate ($c_t' < 0$) up to the inflection point at $q = \frac{b}{6a}$, after which it increases at an increasing rate ($c_t'' > 0$). During the first stage the cost per unit decreases once the initial investment has been spent. During the second stage the cost per unit increases since more capital needs to be invested in order to allow more production capacity.

Definition 44 talks about polynomial, and Example 40 about quadratic equation.

Definition 44. A *polynomial* is an expression in the form $\sum_{i=0}^{n} a_i x^{n-i}$. Here n is called the *order* of the polynomial. If $n = 2$ the polynomial is known as a *quadratic polynomial*, if $n = 3$ a *cubic polynomial*, if $n = 4$ a *quartic*, $n = 5$ a *quintic* and $n = 6$ a *sextic*. If we let $\mathrm{p}(x)$ be a polynomial, then a *polynomial equation* is the equation $\mathrm{p}(x) = 0$. A *polynomial function* is a function of the form $y = \mathrm{f}(x) = \mathrm{p}(x)$.

§

Example 40. The quadratic equation $ax^2 + bx + c = 0$ has the solutions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14}$$

These solutions are called the *roots* of the quadratic equation. Equation 14 is called the 'minus-b formula'. The values of x obtained from the minus-b formula give the intersections of the graph of the quadratic function

$$\mathrm{f}(x) = \mathrm{p}(x) = ax^2 + bx + c$$

on the x-axis. The value $b^2 - 4ac$ determines how the graph of $\mathrm{f}(x)$ lies relative to the x-axis, that is,

$$b^2 - 4ac \begin{cases} > 0, & \text{there are two } x\text{-intersections} \\ = 0, & \text{the graph touches the } x\text{-axis at one point} \\ < 0, & \text{the graph never touches the } x\text{-axis} \end{cases}$$

Furthermore, the graph reverses its direction with respect to the y-axis at the critical point where $\mathrm{f}'(x) = 0$, that is when $x = -\frac{b}{2a}$. Consequently the critical point is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

The graph of $\mathrm{f}(x)$ is symmetric with respect to the vertical line which passes through the turning point, that is to say, the line $x = -\frac{b}{2a}$ The y-intercept is at the point $(0, c)$.

Example 41 is about another form of nonlinear function, the hyperbolic function.

Example 41. A hyperbolic relation is an expression of the form,

$$(px + q)(ry + s) = t$$

From this we obtain,

$$\begin{aligned} \left(x + \frac{q}{p}\right)\left(y + \frac{s}{r}\right) &= \frac{t}{pr} \\ y &= \frac{t}{pr}\left(\frac{1}{x + \frac{q}{p}}\right) - \frac{s}{r} \\ &= \frac{a}{x + b} - c \end{aligned}$$

where p, q, r, s and t are constants, hence so are $a = \frac{t}{pr}$, $b = \frac{q}{p}$, $c = \frac{s}{r}$. In economics we sometimes find hyperbolic functions of the form,

$$y = \frac{a}{bx + c} \tag{15}$$

For example, a demand function of a good may be given by,

$$q + a = \frac{m}{p}$$

which leads to,

$$p = \frac{m}{q + a}$$

where p and q are respectively price and quantity demanded of a good, while m and a are constants.

The graph of Equation 15 has the x-axis, that is the line $y = 0$, as its horizontal asymptote, and has the line $x = -c/b$ as its vertical asymptote. If all the parameters are positive, then the curve in the first quadrant decreases with a decreasing rate.

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Matrix

15^{th} November 2005

Definition 45. Let $A = \{a_{ij}\}$, $B = \{b_{ij}\}$ and $C = \{c_{ij}\}$ be three matrices. Then $C = A+B$ is called the *addition* of the matrices A and B if $c_{ij} = a_{ij}+b_{ij}$ for all i and j.

§

Definition 46. Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{kl})$ an $n \times p$ matrix. Then the product $\mathbf{AB}$ is an $m \times p$ matrix $\mathbf{C} = (c_{il})$ where,

$$c_{il} = \sum_{k=1}^{n} a_{ik} b_{kl}$$

where $1 \le i \le m$ and $1 \le l \le p$.

§

Definition 47. The expression obtained by eliminating the n variables $x_1, \ldots, x_n$ from n equations,

$$\left.\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ &\vdots\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= 0 \end{aligned}\right\} \tag{16}$$

is called the *determinant* of this system of equations, Equation 16. The determinant of matrix A denoted by various different notations, for example $\det(A)$, $|A|$, $\sum(\pm a_1 b_2 c_3 \cdots)$, $D(a_1 b_2 c_3 \cdots)$, or $|a_1 b_2 c_3 \cdots|$.

§

Example 42. For a linear system of three variables, Equation 16 can be written as,

$$\left.\begin{aligned} a_1 x + a_2 y + a_3 z &= 0 \\ b_1 x + b_2 y + b_3 z &= 0 \\ c_1 x + c_2 y + c_3 z &= 0 \end{aligned}\right\} \tag{17}$$

Eliminating x, y and z from Equation 17 gives us,

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_3 b_1 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 - a_3 b_2 c_1 = 0$$

Definition 48. A *minor* M_{ij} of any matrix A is the determinant of a reduced matrix obtained by omitting the i^{th} row and the j^{th} column of A.

§

Theorem 20. Determinant can be determined by,

$$|\,A\,| = \sum_{i=1}^{k} a_{ij} C_{ij}$$

where C_{ij} is called the *cofactor* of a_{ij}. The cofactor C_{ij} can also be denoted as a^{ij}, and,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is a minor of A.

§

Problem 18. Prove Theorem 20,the theorem for finding the determinant of a matrix by Laplacian expansion.

§

Definition 49. Any pairwisely ordered pair in a permutation p is called a *permutation inversion* in p if $i > j$ and $p_i < p_j$.

§

Theorem 21. Determination of the determinant can also be determined by,

$$|\,A\,| = \sum_{\pi} (-1)^{\mathrm{I}(\pi)} \prod_{i=1}^{n} a_{i,\pi(i)}$$

where π is a permutation which ranges over all permutations of $\{1, \ldots, n\}$, and $\mathrm{I}(\pi)$ is called the *inversion number* of π.

§

Problem 19. Prove the theorem for the determination of determinant by permutation, Theorem 21.

§

Theorem 22. Let a be a constant and A an $n \times n$ matrix. Then,

$$|\,aA\,| = a^n\,|\,A\,|$$

$$|\,-A\,| = (-1)^n\,|\,A\,|$$

$$|\,AB\,| = |\,A\,|\,|\,B\,|$$

$$|\,I\,| = |\,AA^{-1}\,| = |\,A\,|\,|\,A^{-1}\,| = 1$$

$$|\,A\,| = \frac{1}{|\,A^{-1}\,|}$$

§

Problem 20. Prove Theorem 22, the theorem on properties of determinant.

§

Definition 50. A function in two or more variables is said to be *multilinear* if it is linear in each variable separately.

§

Theorem 23. Determinants of matrix are multilinear in rows and columns.

§

Example 43. Consider an 3×3 matrix,

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

What Theorem 23 says about multilinearity of determinants amounts to saying that,

$$|A| = \begin{vmatrix} a_1 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

and

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

Problem 21. Prove the theorem on the multilinearity of determinants, Theorem 23.

§

Definition 51. A *conformal mapping* is a transformation that preserves local angle. The terms *function*, *map* and *transformation* are synonyms.

§

Definition 52. A *similarity transformation* is a conformal mapping the transformation matrix of which is,

$$A' \equiv BAB^{-1}$$

Here A and A' are similar matrices.

§

Theorem 24. Similarity transformation does not change the determinant.

Proof. The proof for this is simply,

$$|BAB^{-1}| = |B||A||B^{-1}| = |B||A|\frac{1}{|B|} = |A|$$

¶

Example 44.

$$\begin{aligned} |B^{-1}AB - \lambda I| &= |B^{-1}AB - B^{-1}\lambda IB| \\ &= |B^{-1}(A - \lambda I)B| \\ &= |B^{-1}||A - \lambda I||B| \\ &= |A - \lambda I| \end{aligned}$$

Definition 53. Let A be a square, $n \times n$ matrix. Then the trace of A is,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

§

Definition 54. The *transpose* of a matrix $A = \{a_{ij}\}$ is $A^T = \{a_{ji}\}$.

§

Definition 55. The *complex conjugate* of a matrix $A = \{a_{ij}\}$ is $\bar{A} = \{\bar{a}_{ij}\}$, where $\bar{a} = p - qi$ if $a = p + qi$.

§

Definition 56. Let $\phi(n)$ or $\phi(x)$ be a positive function, and let $f(n)$ or $f(x)$ be any function. Then $f = O(\phi)$ if $|f| < A\phi$ for some constant A and all values of n and x. Here O is called the *big-O* notation which denotes asymptoticity. The notation $f = O(\phi)$ is read, 'f is of order ϕ'.

§

Theorem 25. Some other properties of the determinant are,

$$|\,A\,| = |\,A^T\,|$$

$$|\,\bar{A}\,| = \overline{|\,A\,|}$$

$$|\,I + \epsilon A\,| = 1 + \mathrm{Tr}(A) + O(\epsilon^2), \text{ for } \epsilon \text{ small}$$

§

Example 45. For a square matrix A,

a. switching rows changes the sign of the determinant
b. factoring out scalars from rows and columns leaves the value of the determinant unchanged
c. adding rows and columns together leaves the determinant's value unchanged
d. multiplying a row by a constant c gives the same determinant multiplied by c
e. if a row or a column is zero, then the determinant is zero
f. if any two rows or columns are equal, then the determinant is zero

Problem 22. Prove the properties of determinant given in Theorem 25.

§

Theorem 26. Some properties of matrix trace are,

$$\mathrm{Tr}(A) = \mathrm{Tr}\left(A^T\right)$$

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

$$\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$$

§

Problem 23. Prove that,

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

§

Theorem 27.

$$(AB)^T = B^T A^T$$

Proof.

$$\begin{aligned}\left(B^T A^T\right)_{ij} &= \left(b^T\right)_{ik}\left(a^T\right)_{kj}\\ &= b_{ki}a_{jk}\\ &= a_{jk}b_{ki} = (AB)_{ji} = (AB)^T_{ij}\end{aligned}$$

¶

Definition 57. Let A be a square matrix. Then the *inverse* of A, if it exists, is A^{-1} such that,

$$AA^{-1} = I$$

Furthermore, A is said to be *nonsingular* or *invertible* if its inverse exists, otherwise it is said to be *singular*.

§

Example 46. For a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse of A is,

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If A is a 3×3 matrix, then the inverse of A is,

$$A^{-1} = \frac{1}{|A|}\left\{\det\left(m_{ij}\right)\right\}$$

where m_{ij} is a minor of A.

If A is an $n \times n$ matrix, then A^{-1} can be found by numerical methods, for example Gauss-Jordan elimination, Gaussian elimination, and LU decomposition.

Example 47. The *Gaussian elimination* procedure solves the matrix equation $A\mathbf{x} = \mathbf{b}$ by first forming an augmented matrix equation $[A\,\mathbf{b}]$ and then transform this into an upper triangular matrix $\left[\{a'_{ij}\}\,\mathbf{b}'\right]$, where a'_{ij} are all zero except when $i \leq j$. Then,

$$x_i = \frac{1}{a'_{ii}}\left(b'_i - \sum_{j=i+1}^{k} a'_{ij}x_j\right)$$

The *Gauss-Jordan elimination* procedure finds matrix inverse by first forming a matrix $[A\,I]$, and then use the Gaussian elimination to transform this matrix into $[I\,B]$. The result matrix B is in fact A^{-1}.

The *LU decomposition* forms from the matrix A a product LU of two matrices, one lower- while the other upper triangular. This gives us three types of equation to solve, namely when $i < j$, $i = j$ and $i > j$, where i and j are the indices of row and respectively column of the matrix product. Then,

$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$$

Letting $\mathbf{y} = U\mathbf{x}$ we have $L\mathbf{y} = \mathbf{b}$, and therefore,

$$\begin{aligned} y_1 &= \frac{b_1}{l_{11}} \\ y_i &= \frac{y}{l_{ii}}\left(b_i - \sum_{j=1}^{i-1} l_{ij}y_j\right) \end{aligned}$$

where $i = 2, \ldots, n$. Then since $U\mathbf{x} = \mathbf{y}$,

$$\begin{aligned} x_n &= \frac{y_n}{u_{nn}} \\ x_i &= \frac{1}{n_{ii}}\left(y_i - \sum_{j=i+1}^{n} u_{ij}x_j\right) \end{aligned}$$

where $i = n-1, \ldots, 1$.

Theorem 28. Let A and B be two square matrices of equal size. Then,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. Let $C = AB$. Then $B = A^{-1}C$ and $A = CB^{-1}$, therefore,

$$C = AB = \left(CB^{-1}\right)\left(A^{-1}C\right) = CB^{-1}A^{-1}C$$

Hence $CB^{-1}A^{-1} = I$, and thus $B^{-1}A^{-1} = (AB)^{-1}$. ¶

Definition 58. The *Einstein's summation* is the simplification of notation by omitting a summation sign, keeping in mind that repeated indices are implicitly summed over, for example $\sum_i a_{ik}a_{ij}$ becomes $a_{ik}a_{ij}$, and $\sum_i a_i a_i$ becomes $a_i a_i$.

§

Definition 59. The multiplication of two matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ is the matrix $C = AB$ such that $c_{ik} = a_{ij}b_{jk}$.

§

Theorem 29. The matrix multiplication is associative.

Proof.

$$[(ab)c]_{ij} = (ab)_{ik}c_{kj} = (a_{il}b_{lk})\,c_{kj} = a_{il}\,(b_{lk}c_{kj}) = a_{il}(bc)_{lj} = [a(bc)]_{ij}$$

¶

Example 48. From Theorem 29, which shows us the associativity of matrix multiplication, we could write the multiplication of three matrices as $[abc]_{ij}$, which is the same as writing $a_{il}b_{lk}c_{kj}$. And this applies in a similar manner to the multiplication of four or more matrices.

Theorem 30. If A and B are two square and diagonal matrices, then $AB = BA$. But in general matrix multiplication is not commutative.

§

Problem 24. Prove Theorem 30, which is a theorem about non-commutativity of matrix multiplication.

§

Definition 60. A *block matrix* is a matrix which is is made up of small matrices put together, for example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are matrices.

§

Theorem 31. Block matrices may be multiplied together in the usual manner, for example,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}\begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix}$$

provided that all the products involved are possible.

§

Problem 25. Prove Theorem 31.

§

Definition 61. Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Then A is called a *diagonal matrix* if $a_{ij} = 0$ when $i \neq j$. Here $1 \leq i, j \leq n$. In other words, a diagonal matrix has its components in the form $a_{ij} = c_i\delta_{ij}$, where c_i is a constant and δ_{ij} is the Kronecker delta,

$$\delta = \begin{cases} 0, \, i \neq j \\ 1, \, i = j \end{cases}$$

§

Theorem 32. A square matrix A can be diagonalised by the transformation $A = PDP^{-1}$, where P is made up of the eigenvectors of A and D is the diagonal matrix desired.

§

Problem 26. Prove Theorem 32, the theorem on matrix diagonalisation.

§

Example 49. Matrix diagonalisation can greatly help reducing the number of parameters in a system of equations. For instance, the systems $A\mathbf{x} = \mathbf{y}$ when diagonalised becomes $PDP^{-1}\mathbf{x} = \mathbf{y}$, that is $D\mathbf{x}' = \mathbf{y}'$, where $\mathbf{x}' = P^{-1}\mathbf{x}$ and $\mathbf{y}' = P^{-1}\mathbf{y}$. In this case, if A is an $n \times n$ matrix, we say that our new system obtained through the process of diagonalisation has canonicalised from $n \times n$ to n parameters.

Definition 62. A *symmetric* matrix is a square matrix A which satisfies $A^T = A$.

§

Example 50. If A is a symmetric matrix, then $A^{-1}A^T = I$.

Definition 63. Let A be a square matrix. Then A is said to be *orthogonal* if $AA^T = I$.

§

Example 51. Definition 63 is the same as saying that $A^{-1} = A^T$.

Theorem 33. A matrix A is symmetric if it can be expressed as $A = QDQ^T$, where Q is an orthogonal matrix and D is a diagonal matrix.

§

Problem 27. Prove Theorem 33, the problem on symmetric matrix.

§

Example 52. Any square matrix A may be decomposed into two terms added together, that is $A_s + A_a$ where A_s is a symmetric matrix and A_a an antisymmetric matrix, called respectively a *symmetric part* and an *antisymmetric part* of A. Furthermore,

$$A_s = \frac{1}{2}\left(A + A^T\right)$$

and,

$$A_a = \frac{1}{2}\left(A - A^T\right)$$

Linear algebra

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Definition 64. Let A be a square, nonsingular matrix. Then the *inverse matrix* A^{-1} of A is a unique matrix for which,

$$AA^{-1} = I = A^{-1}A$$

§

Example 53. An inverse matrix may be found using the formula,

$$A^{-1} = \frac{1}{|\,A\,|}\operatorname{Adj} A$$

Example 54. Matrix equations of the form $A\mathbf{x} = \mathbf{b}$ can be solved with the help of the inverse matrix A^{-1} as $\mathbf{x} = A^{-1}\mathbf{b}$, where A is an $n \times n$ matrix, $\mathbf{x}$ a vector of size n whose components are variables, and $\mathbf{b}$ a vector of size n containing constants.

Theorem 34. Let A be the coefficient matrix and A_i a matrix formed from A by replacing the column of coefficients of x_i with the column vector of constants. Cramer's rule solves a system of linear equations through the use of determinants as follows.

$$x_i = \frac{|\,A_i\,|}{|\,A\,|}$$

§

Problem 28. Prove Cramer's rule, Theorem 34.

§

Definition 65. Let a system of n functions not necessarily linear be

$$\begin{aligned} y_1 &= f_1\,(x_1, \ldots, x_n) \\ &\vdots \\ y_n &= f_n\,(x_1, \ldots, x_n) \end{aligned}$$

Then a *Jacobian determinant* comprises all the first-order partial derivatives of the system arranged in ordered sequence, that is

$$|\,J\,| = \left|\,\frac{\partial y_1, \ldots, \partial y_n}{\partial x_1, \ldots, \partial x_n}\,\right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

§

Theorem 35. Let a system of n equations be $y_i = f_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$. If $|\, J \,| = 0$, then y_i are functionally dependent. On the other hand if $|\, J \,| \neq 0$, then y_i are functionally independent.

§

Problem 29. Prove Theorem 35.

§

Definition 66. A determinant $|\, H \,|$ composed of all the second-order partial derivatives, with the direct partials on the principal diagonal and the cross partials off the same, is called a *Hessian.* In other words, let a multivariable function be $z = f(x, y)$. Then the Hessian of z is

$$|\, H \,| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix}$$

where $z_{xy} = z_{yx}$. Moreover, the *first principal minor* is $|\, H_1 \,| = z_{xx}$ and the *second principal minor* is

$$|\, H_2 \,| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} = z_{xx} z_{yy} - (z_{xy})^2$$

§

Theorem 36. Let a multivariable function be $z = f(x, y)$, and let the first-order conditions $z_x = z_y = 0$ are met. Then a sufficient condition for z to be at optimum is $z_{xx} z_{yy} > (z_{xy})^2$ together with $z_{xx}, z_{yy} < 0$ in case of a maximum and $z_{xx}, z_{yy} > 0$ in case of a minimum.

§

Problem 30. Prove Theorem 36, the theorem on the optimality of a multivariable function.

§

Definition 67. From Definition 66, if $|\, H_1 \,| > 0$ and $|\, H_2 \,| > 0$ the Hessian $|\, H \,|$ is said to be *positive definite*, and the second-order conditions for the minimum are met. If $|\, H_1 \,| < 0$ and $|\, H_2 \,| > 0$ it is said to be *negative definite*, and the second-order conditions for the maximum are met.

§

Algorithm 1 *Procedure to test for the optimality of multivariable functions of two variables.*

```
$z = f(x, y)$
find $z_x$ and $z_y$
if $z_x = 0$ and $z_y = 0$ then
  find $z_{xx}$, $z_{xy}$ and $z_{yy}$
  find $H_1$ and $H_2$
  if $|\, H_1 \,| > 0$ and $|\, H_2 \,| > 0$ then
```

```
        $|H|$ is positive definite
    elseif $|H_1| < 0$ and $|H_2 > 0|$ then
        $|H|$ is negative definite
    endif
endif
```

Definition 68. Let $y = f(x_1, \ldots, x_n)$ be function of n variables. Then the n^{th}-order Hessian for this function is

$$|H| = \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix}$$

Then the *first principal minor* $|H_1|$ is simply x_{11}, and the i^{th} principal minor is

$$|H_i| = \begin{vmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & \ddots & \vdots \\ y_{i1} & \cdots & y_{ii} \end{vmatrix}$$

§

Theorem 37. Let $y = f(x_1, \ldots, x_n)$ be function of n variables. Let the Hessian of y be represented by $|H|$. Then if all the principal minors of $|H|$ are positive, then $|H|$ is positive definite and the second-order conditions for a relative minimum are met. If the sign of the principal minors alternates between negagive and positive, then $|H|$ is negative definite and the second-order conditions for a relative maximum are met.

§

Example 55. For $y = f(x_1, x_2, x_3)$ the third-order Hessian is

$$|H| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

where

$$y_{11} = \frac{\partial^2 y}{\partial x_1^2}, \quad y_{12} = \frac{\partial^2 y}{\partial x_2 \partial x_1}, \quad \text{and so on}$$

The first-, second- and third-order Hessian's are respectively

$$|H_1| = y_{11}, \quad |H_2| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} \quad \text{and} \quad |H_3| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

If $|H_1| > 0$, $|H_2| > 0$ and $|H_3| > 0$, then H is positive definite and the second-order condition for minimum is fulfilled. If $|H_1| < 0$, $|H_2| > 0$ and

$|H_3| < 0$, then $|H|$ is negative definite and the second-order condition for maximum is satisfied.

Definition 69. A *discriminant* is a determinant of a quadratic form. Let the quadratic form be $z = ax^2 + bxy + cy^2$, which is in matrix form

$$z = [\,x \quad y\,]\begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then the discriminant is

$$|D| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

The *first principal minor* of the discriminant is $|D_1| = a$, and the *second principal minor*

$$|D_2| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} = ac - \frac{b^2}{4}$$

§

Theorem 38. Let a quadratic form be

$$z = ax^2 + bxy + cy^2$$

and let the discriminant of z be $|D|$. If $|D_1| > 0$ and $|D_2| > 0$, then $|D|$ is positive definite and $z > 0$ for all $x, y \neq 0$. If $|D_1| < 0$ and $|D_2| > 0$, then $|D|$ is negative definite and $z < 0$ for all $x, y \neq 0$.

§

Theorem 39. Let $f(x, y)$ be a function subject to a constraint $g(x, y) = k$, where k is a constant. Then the optimisation of f can be done by first transforming f together with g into a new function

$$F(x, y, \lambda) = f(x, y) + \lambda\,(k - g(x, y))$$

and then solve the following equations,

$$\begin{aligned} F_x(x, y, \lambda) &= 0 \\ F_y(x, y, \lambda) &= 0 \\ F_z(x, y, \lambda) &= 0 \end{aligned}$$

to obtain the critical values x_0, y_0 and λ_0 at which F and hence f are optimised.

§

Problem 31. Prove Theorem 38, the definiteness of a function by the discriminant. Prove the theorem for constrained optimisation with Lagrange multipliers, Theorem 39 above.

§

Definition 70. In the constrained optimisation with Lagrange multipliers in Theorem 39 above, f is called an *objective* or *origin function* and F the Lagrangian function.

§

Definition 71. Let $f(x_1, \ldots, x_n)$ be a function of n variables subject to constraints $g(x_1, \ldots, x_n)$. Let

$$F(x_1, \ldots, x_n, \lambda) = f(x, \ldots, x_n) + \lambda\,(k - g(x_1, \ldots, x_n))$$

Then the *bordered Hessian* $|\,\bar{H}\,|$ is defined as either

$$|\,\bar{H}\,| = \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & g_1 \\ F_{21} & & & & g_2 \\ \vdots & & \ddots & & \vdots \\ F_{n1} & & & F_{nn} & g_n \\ g_1 & g_2 & \cdots & g_n & 0 \end{vmatrix}$$

or

$$|\,\bar{H}\,| = \begin{vmatrix} 0 & g_1 & \cdots & g_n \\ g_1 & F_{11} & & F_{1n} \\ \vdots & & \ddots & \vdots \\ g_n & F_{n1} & \cdots & F_{nn} \end{vmatrix}$$

This is simply the Hessian

$$\begin{vmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & \ddots & \vdots \\ F_{n1} & \cdots & F_{nn} \end{vmatrix}$$

bordered by the first derivatives of the constraint with zero on the principal diagonal. The order of a bordered principal minor being determined by the order of the principal minor being bordered, $|\,\bar{H}\,| = |\,\bar{H}_n\,|$ since in this case an $n \times n$ principal minor is being bordered.

§

Theorem 40. Let $f(x_1, \ldots, x_n)$ be a function of n variables subject to constraints $g(x_1, \ldots, x_n)$. Let $|\,\bar{H}\,|$ be the bordered Hessian defined in Definition 71. Then if $|\,\bar{H}_2\,|, \ldots, |\,\bar{H}_n\,| < 0$, then the bordered Hessian $|\,\bar{H}\,|$ is positive definite, and therefore is a sufficient condition for a minimum. If $|\,\bar{H}_2\,| > 0$ $|\,\bar{H}_3\,| < 0$, $|\,\bar{H}_4\,| > 0$, and so alternatingly on, then $|\,\bar{H}\,|$ is negative definite, which is a sufficient condition for a maximum.

§

Problem 32. Prove Theorem 40.

§

Example 56. Let $f(x,y)$ be a function to be optimised subject to a constraint $g(x,y) = k$, where k is a constant. Then the Lagrangian function becomes

$$F(x,y,\lambda) = f(x,y) + \lambda\,(k - g(x,y))$$

The first-order conditions for optimisation are $F_x = F_y = F_\lambda = 0$. The second-order conditions for optimisation can be expressed together as a bordered Hessian

$$|\bar{H}| = \begin{vmatrix} F_{xx} & F_{xy} & g_x \\ F_{yx} & F_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$$

or

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

Note 1. Theorem 39 gives the first-order conditions for optimising a function subject to some constraints. Theorem 40 gives the second-order conditions for optimising a function subject to some constraints.

§

Definition 72. A *Marshallian demand function* gives an expression of the amount of a good that a consumer will buy as a function of commodity prices and income available. It is derived by maximising the utility subjected to a budgetary constraint.

§

Example 57. Let a utility be $u = q_1q_2$ which is subject to a constraint $p_1q_1 + p_2q_2 = b$, where b is the amount of income available, that is to say, our budget. Then the Lagrangian function is

$$U = q_1q_2 + \lambda\,(b - p_1q_1 - p_2q_2)$$

The first partial derivatives are then

$$u_1 = q_2 - \lambda p_1 = 0 \tag{18}$$
$$u_2 = q_1 - \lambda p_2 = 0 \tag{19}$$
$$u_\lambda = b - p_1q_1 - p_2q_2 = 0 \tag{20}$$

where u_1, u_2 are respectively u_{q_1} and u_{q_2}. Simultaneously solving Equation's 18, 19 and 20 leads us to

$$\frac{q_2}{p_1} = \lambda = \frac{q_1}{p_2}$$

Hence $q_2 = q_1p_1/p_2$ and $q_1 = q_2p_2/p_1$ and from Equation 20 we have,

$$b = p_1q_1 + p_2\frac{p_1q_1}{p_2} = p_2q_2 + p_1\frac{p_2q_2}{p_1}$$

which yield us, for q_1 and q_2, the Marshallian demand functions which maximise satisfaction of the consumer subject to income and prices.

Next, we test the second-order conditions by firstly finding $u_{11} = 0$, $u_{22} = 0$, $u_{12} = u_{21} = 1$, $g_1 = p_1$ and $g_2 = p_2$, which give us

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & p_1 \\ 1 & 0 & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}$$

which gives $|\bar{H}_2| = 2p_1p_2 > 0$ Hence $|\bar{H}|$ is negative definite and thus u is maximised.

Definition 73. The production process of producing one good usually requires the input of many other *intermediate goods.* Let x_i be the total demand for product i, and let b be the final demand for the product from the ultimate users. Then,

$$x_i = a_{i1}x_1 + \ldots + a_{in}x_n + b_i$$

for $i = 1, \ldots, n$, where a_{ij} is a *technical coefficient* which represents the value of input i required to produce one monetary unit's worth of product j. If we consider the total demand for every one of the products, then

$$\mathbf{x} = A\mathbf{x} + \mathbf{b}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

It follows from this that $\mathbf{x} = (I - A)^{-1}\mathbf{b}$. The matrix A is known as the *matrix of technical coefficients.* It is also known as the *input-output table*, the rows being the inputs and the columns the outputs.. The matrix $I - A$ is known as the *Leontief matrix.*

§

Example 58. In a complete input-output table, labour and capital would also be included as inputs. These give the value added by the firm. They are normally put as an extra row at the bottom of the matrix of technical coefficients A. The vertical summation of each column of the table is then equal to 1.

Definition 74. Let A be a square matrix. Then a scalar λ such that the equation

$$A\mathbf{v} = \lambda\mathbf{v} \tag{21}$$

holds for some vector $\mathbf{v} \neq \mathbf{0}$ is called an *eigenvalue* † of A, and the vector $\mathbf{v}$ is called an *eigenvector* of A corresponding to the eigenvalue λ. The eigenvalue λ is also known as the *characteristic root*, or the *latent root*, while the eigenvector is also known as the *characteristic vector*, or the *latent vector*.

§

Note 2. From Equation 21 it follows directly that

$$(A - \lambda I)\mathbf{v} = 0 \tag{22}$$

Then $A - \lambda I$ is called the *characteristic matrix* of A. Since $\mathbf{v}$ assumes a unique value and by assumption $\mathbf{v} \neq 0$, it follows that $A - \lambda I$ must be singular, which means that its rows must be a multiple of one another. Now $A - \lambda I$ is zero if and only if the *characteristic determinant* $|\,A - \lambda I\,|$ of A is zero. In other words

$$|\,A - \lambda I\,| = 0 \tag{23}$$

which is called the *characteristic equation* of A. With Equation 23 there will be an infinite solution for $\mathbf{v}$ in Equation 22. In particular, if $\mathbf{v}$ is a solution, that is if it is an eigenvector, so is $k\mathbf{v}$ for any $k \neq 0$. We force a unique solution by using the *normalisation*

$$\sum v_i^2 = 1$$

Then the sign-definiteness of A can be determined from the characteristic roots λ's. Thus if all λ's are positive, then A is positive definite; and if negative, negative definite. Let at least one λ be zero, which is neither positive nor negative, if all the remaining λ's are nonnegative, then A is positive semidefinite; and if they are nonpositive, negative semidefinite. Lastly, if some of the λ's are positive while others are negative, then A is indefinite.

§

Problem 33. Prove all the necessary details need to be proved in Note 2.

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Note 3. We have seen in Note 2 how, having found λ_i, where $i = 1, \ldots, n$, we find through normalisation the corresponding, unique $\mathbf{v}_i$. On the other hand if we have found first the $\mathbf{v}_i$'s, their corresponding λ_i's may be found by first forming a *transformation matrix*

$$T = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$$

---

† The word *eigenvalue* is a half-translation of the German word *Eigenwert*. The latter means ‘appropriate value’ since *Wert* means ‘value’ and *eigen* means ‘proper or appropriate’.

and then the corresponding eigenvalues or the characteristic roots are obtained from

$$T^T A T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \lambda_n \end{bmatrix}$$

§

Problem 34. Prove all the necessary details in Note 3.

§

Definition 75. The vector equation, Equation 21, has as its solutions the zero vector $\mathbf{v} = 0$ together with all the corresponding eigenvalue-eigenvector pairs. The set of all the eigenvalues of A is called the *spectrum* of A. The *spectral radius* of A is then the largest of all the absolute values of the eigenvalues of A, that is to say,

$$\max_i |\lambda_i|$$

The set of all eigenvectors $\mathbf{v}_{ij}$, together with $\mathbf{0}$, forms a vector space called the *eigenspace* of A corresponding to λ_i.

§

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Examples for linear algebra

14^{th} January, 2007

1. Solve using the Gaussian elimination,

$$\begin{aligned} 3x_1 + 8x_2 &= 53 \\ 6x_1 + 2x_2 &= 50 \end{aligned}$$

Solution. Write the equations in an augmented matrix,

$$[A \quad B] = \begin{bmatrix} 3 & 8 & 53 \\ 6 & 2 & 50 \end{bmatrix}$$

Row 1 times $\frac{1}{3}$,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 6 & 2 & 50 \end{bmatrix}$$

Row 2 subtracted by 6 times of Row 1,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 0 & -14 & -56 \end{bmatrix}$$

Row 2 times $-\frac{1}{14}$,

$$\begin{bmatrix} 1 & \frac{8}{3} & \frac{53}{3} \\ 0 & 1 & 4 \end{bmatrix}$$

Row 1 subtracted by $\frac{8}{3}$ times of Row 2,

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 4 \end{bmatrix}$$

Therefore $\bar{x}_1 = 7$ and $\bar{x}_2 = 4$.

#

2. Let

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

Find all the eigenvalues and a basis of each eigenspace. Can A be diagonalised? Why?

Solution. Form the characteristic matrix $tI - A$ and find its determinant to obtain the characteristic polynomial $\Delta(t)$ of A;

$$\Delta(t) = |\,tI - A\,| = \begin{vmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{vmatrix} = (t+2)^2(t-4)$$

The roots of $\Delta(t)$ are -2 and 4, hence the eigenvalues of A are -2 and 4.

#

Next, find a basis of the eigenspace of the eigenvalue -2. Substitute $t=-2$ into the characteristic matrix $tI-A$, thus obtain the homogeneous system

$$\begin{pmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In other words,

$$\begin{cases} -3x+3y-3z=0 \\ -3x+3y-3z=0 \\ -6x+6y-6z=0 \end{cases}$$

that is $x-y+z=0$. The system has two independent solutions, for example $x=1$, $y=1$, $z=0$ and $x=1$, $y=0$, $z=-1$. Therefore $u=(1,1,0)$ and $v=(1,0,-1)$ are independent eigenvectors which generate the eigenspace of the eigenvalue -2. In other words, u and v form a basis of the eigenspace of -2. This means that every eigenvector belonging to -2 is a linear combination of u and v.

Similarly, find a basis of the eigenspace of the eigenvalue 4. Substitute $t=4$ into the characteristic matrix $tI-A$ to obtain the homogeneous system

$$\begin{pmatrix} 3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This is the same as writing

$$\begin{cases} 3x+3y-3z=0 \\ -3x+9y-3z=0 \\ -6x+6y=0 \end{cases}$$

which can be reduced to

$$\begin{cases} x+y-z=0 \\ 2y-z=0 \end{cases}$$

Since the number of independent equations is less than the number of variables by one, this system has only one free variable. Therefore any particular non-zero solution, for example $x=1$, $y=1$, $z=2$ generates its solution space. Hence $w=(1,1,2)$ is an eigenvector which generates, and so form a basis of the eigenspace of the eigenvalue 4.

Since A has three linearly independent eigenvectors, A is diagonalisable.

#

Let P be the matrix the columns of which are the three independent eigenvectors,

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

Then

$$P^{-1}AP = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The diagonal elements of $P^{-1}AP$ are the eigenvalues of A corresponding to the columns of P.

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Exercises for linear algebra

14^{th} January, 2007

3. Find values of the variables x, y and z for each of the following systems of equations, using Gaussian elimination.

$$(i)\quad \begin{cases} 2x-2y-z &= 3\\ x-y+z &= 2\\ x+y+2z &= 3 \end{cases} \qquad (ii)\quad \begin{cases} 4x+3y+z &= 5\\ 2x-y-z &= 4\\ x+y-z &= 3 \end{cases}$$

$$(iii)\quad \begin{cases} 2x+2y-7z &= 10\\ -x-y &= 5\\ 3x+2y+z &= -1 \end{cases} \qquad (iv)\quad \begin{cases} 3x+2y+z &= 16\\ x+y+z &= 5\\ 2x-y-z &= 7 \end{cases}$$

Linear programming

29th November 2005

Definition 76. A problem of *optimisation* is one in which one tries to maximise or minimise a certain quantity called the *objective*, which depends on a finite number of variables. These may be either independent or related to one another through some *constraints.*

§

Definition 77. A *mathematical programme* is an optimisation problem in which the objective and the constraints are given as functions or mathematical relationship. In other words,

$$\begin{aligned} \text{optimise:} \quad & z = f(x_1, \ldots, x_n) \\ \text{subject to:} \quad & g_i(x_1, \ldots, x_n) \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} b_i, \quad i = 1, \ldots, m \end{aligned}$$

Some constraints are explicitly stated as requirements, others are hidden conditions. These latter need to be pin-pointed through the study and understanding of the model and its inputs.

§

Definition 78. A *linear programme* is a mathematical programme all the functions involved of which are linear. This means that,

$$\begin{aligned} f(x_1, \ldots, x_n) &= c_1 x_1 + \cdots + c_n x_n \\ g_i(x_1, \ldots, x_n) &= a_{i1} x_1 + \cdots + a_{in} x_n \end{aligned}$$

where $i = 1, \ldots, m$ and c_j and a_{ij}, $j = 1, \ldots, n$, are constants. If there is an additional restriction on the input variables that they be all integers, then the optimisation problem is called an *integer programme.* A mathematical programme which is not a linear programme is said to be *nonlinear.*

§

Definition 79. A *quadratic programme* is a mathematical programme in which all the constraints are linear and the objective function is in quadratic form, which is in general,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_i x_j + \sum_{i=1}^{n} d_i x_i$$

where c_{ij} and d_i are constants.

§

Definition 80. A linear programme is said to be in *standard form* if all the constraints are equalities and if one feasible solution is known. In other words, our problem is now

$$\begin{aligned}\text{optimise:}\quad & z = \mathbf{c}^T\mathbf{x}\\ \text{subject to:}\quad & A\mathbf{x} = \mathbf{b}\\ \text{with:}\quad & \mathbf{x} \geq 0\end{aligned}$$

§

Definition 81. One may change any linear programme into the standard form by adding a *slack variable* to the left-hand side of a constraint of the form $\sum a_{ij}x_j \leq b_i$ to obtain

$$\sum_{j=1}^{n} a_{ij}x_j + x_{p_k} = b_i$$

where $p_k > n$ and $k = 1, 2, \ldots$. Similarly one may add a *surplus variable* to the right-hand side of a constraint of the form $\sum a_{ij}x_j \geq b_i$ to obtain $\sum a_{ij}x_j = b_i + x_{q_l}$

$$\sum_{j=1}^{n} a_{ij}x_j - x_{q_l} = b_i$$

where $q_l > n$ and $l = 1, 2, \ldots$. Next, all the slack and surplus variables are added to the objective function with zero coefficients. Then if we add an *artificial variable* to the left-hand side of each constraint where there is no slack variable, then the *initial feasible solution* is $\mathbf{x}_0 = \mathbf{b}$, where $\mathbf{x}$ is the vector of slack and artificial variables. The artificial variables are added to the objective function with a large negative coefficient $-M$.

§

Definition 82. A set of n vectors of m dimensions $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist some constants $\alpha_1, \ldots, \alpha_n$ not all of which are zero, such that

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \qquad (24)$$

It is said to be *linearly independent* if the condition in Equation 24 implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

§

Theorem 41. Consider a set of n vectors of m dimensions. If $n > m$, then the set is linearly dependent.

§

Problem 35. Prove Theorem 41.

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Definition 83. A vector $\mathbf{v}$ is called a *convex combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if there exist some nonnegative constants $\beta_1, \ldots, \beta_n$, where

$$\beta_1 + \cdots + \beta_n = 1$$

such that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$$

§

Definition 84. A set of m-dimensional vectors is said to be *convex* if for any two vectors belonging to the set the line segment between them also belongs to the set.

§

Theorem 42. All points on the line segment joining any two vectors may be expressed as a convex combination of the two vectors.

§

Problem 36. Prove Theorem 42.

§

Definition 85. A vector $\mathbf{v}$ is called an *extreme point* of a convex set if it can not be expressed as a convex combination of two other vectors in the set.

In other words, an extreme point of a convex set K is a point x in K that cannot be written as $x = \theta y + (1 - \theta) z$ with $0 < \theta < 1$, y and z in K, and $y \neq z$, that is to say, an extreme point is a point which is not an *interior point* of any line segment belonging to K.
An equivalent definition of an extreme point is that x is an extreme point of a convex set K if $K \setminus \{x\}$ is convex.

§

Definition 86. A *metric space* is a non-empty set X for which is defined a concept of distance. The *distance* d is called a *metric* on X, having such properties that, for any points x and y in X, we have $d(x, y) \geq 0$, and $d(x, y) = 0$ implies $x = y$; $d(x, y) = d(y, x)$; and $d(x, y) \leq d(x, z) + d(z, y)$.

Let X be a metric space with metric d, let A be a subset of X and let x be any point of X. Then the *distance from x to A* is defined as

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

whereas the *diameter of A* is defined as

$$d(A) = \sup \{d(a_1, a_2) : a_1 \,\text{and}\, a_2 \in A\}$$

Then a set is said to be *bounded* if its diameter is finite.

Further, let x_0 be a point in X and r a positive real number. Then the *open sphere* $S_r(x_0)$ with *centre* x_0 and *radius* r is the subset of X defined by

$$S_r(x_0) = \{x : d(x, x_0) < r\}$$

A point x in X is called a *limit point* of A if each open sphere centred on x contains at least one point of A different from x. A subset F of X is said to be *closed* if it contains all its limit points.

§

Definition 87. A *linear space*, aka a *vector space*, is a non-empty set L on which is defined two binary processes, say *addition* and *scalar multiplication.* Addition is defined such that for any x, y and z in L, then $x+y$ is again in L; $x+y=y+x$; $x+(y+z)=(x+y)+z$; there exists a unique *identity* element 0, aka *zero element* or the *origin*, such that $x+0=x$ for every x; and there exists a unique *inverse* element $-x$ for every x, such that $x+(-x)=0$. Scalar multiplication is defined with regard to *scalars*, some instances of which are real and complex numbers, such that for any scalar α and any x and y in L, αx is again in L; $\alpha(x+y)=\alpha x+\alpha y$; $(\alpha+\beta)x=\alpha x+\beta x$; $(\alpha\beta)x=\alpha(\beta x)$; and $1x=x$, where 1 is the identity for scalar multiplication. A *normed linear space* is a linear space on which is defined a *norm*, that is a function which maps each element x in the space to a real number $||x||$ in such a manner that $||x|| \geq 0$, and $||x||=0$ if and only if $x=0$; $||x+y|| \leq ||x||+||y||$; and $||\alpha x|| = |\alpha|\,||x||$.

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Theorem 43. A normed linear space is a metric space.

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Problem 37. Prove Theorem 43.

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Theorem 44. Any vector in a closed and bounded convex set with a finite number of extreme points can be expressed as a convex combinations of the extreme points.

§

Problem 38. Prove Theorem 44.

§

Definition 88. For two vectors, that is points, x and y in $\mathbf{R}^n$, we write $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for all $1 \leq i \leq n$. A system of m weak linear inequalilties in n variables can be written as $A\mathbf{x} \geq \mathbf{b}$, where A is an $m \times n$ matrix. A fundamental question concerning such system is whether it is *consistent*, that is to say, whether there exists some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. A system may be *inconsistent*, or it may have a set of solutions which is *unbounded.* If we sketch our problem on a graph, we may see that it's solution set is *convex.*

§

Theorem 45. The solution space of a set of simultaneous linear equations is a convex set the number of extreme points of which is finite.

§

Problem 39. Prove Theorem 45.

§

Theorem 46. Let S be the set of all feasible solutions to the linear programme in standard form in Definition 80, in other words, S is the set of all vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where A is an $m \times n$ matrix. Then S is a convex set, and the number of its extreme points is finite. The objective function attains its optimum, provided that one exists, at an extreme point of S. If $m \leq n$, then the extreme points of S have at least $n - m$ zero components.

§

Problem 40. Prove Theorem 46.

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Algorithm 2 *Procedure for finding basic feasible solutions.*

```
Input: $A\mathbf{x} = \mathbf{b}$, $A$ is an $m \times n$ matrix, $m \leq n$, $\operatorname{rank} A = m$
$[\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] \leftarrow A$
$(x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}) \leftarrow (A\mathbf{x} = \mathbf{b})$
for $i = m + 1$ to $n$ do
    $x_i \leftarrow 0$
endfor
$(x_1, \ldots, x_n) \leftarrow$ solve $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$
```

Definition 89. The *simplex method* is a matrix procedure which solves linear programmes of the standard form as described in Definition 80 where $\mathbf{b} > \mathbf{0}$. Starting from a basic feasible solution $\mathbf{x}_0$ we locate successively other basic feasible solutions giving better values for our objective. For minimisation programmes the method uses Table 3, for maximisation programmes the same table is also used but with the sign of entries in the bottom row reversed.

§

Table 3 *Table used for minimisation programming in simplex method.*

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{c}^T - \mathbf{c}_0^T A$	$-\mathbf{c}_0^T\mathbf{b}$

§

Table 4 *Description of the simplex method.*

while negative number exists in **d do**

Locate the most negative number in the bottom row of the simplex table, excluding the last column. The column in which we find this number is called the *work column*. If more than one such column exist, choose one of them.

Find the smallest of the ratios between the elements in the last column and the elements in the work column of the same row, if these latter are positive. The element in the work column that yields this smallest ratio is called the *pivot element*. If there are more than one of these, choose one. If none of the elements in the work column is positive, the programme has no solution.

Using elementary row operations, convert the pivot element to 1 and reduce all other elements in the work column to 0.

Replace the x-variable in the pivot row and first column by the x-variable in the first row and pivot column. This new first column then becomes the current set of basic variables.

endwhile

The optimal solution is one in which all the basic variables assume the corresponding values in the last column, while the remaining variables are zero. The optimal value of the objective function is then the value of the last row and last column for a maximisation programme, and the negative of this value if the programme is one of minimisation.

§

Algorithm 3 gives a procedure for the simplex method. Here M represents a large positive integer, ρ a ratio, c a column, r a row, and $\mathbf{x}_0$ contains all the basic variables. Also the last row in Table 3 is represented here by $\mathbf{d} = \mathbf{c}^T - \mathbf{c}_0^T A$ and $e = -\mathbf{c}_0^T \mathbf{b}$.

Algorithm 3 *Algorithm for the simplex procedure.*

```
$j \leftarrow 0$
while there exists a negative number in $\mathbf{d}$ do
  $j \leftarrow j + 1$
  for $i = 1$ to $n$ do
    $\{c\text{'s}\} \leftarrow$ (column number of the most negative number in the bottom row)
    (work column) $\leftarrow$ choose one of the $\{c\}$'s
    $k \leftarrow$ (work column)
  endfor
  $\rho_{pivot} \leftarrow M$
  $c \leftarrow 0$
  $soln \leftarrow 0$
  for $i = 1$ to $m$ do
    if $(\mathbf{a}_j)_{ik} > 0$ then
      $soln \leftarrow 1$
```

```
            $\rho \leftarrow \frac{(\mathbf{b}_j)_i}{(\mathbf{a}_j)_{ik}}$
            if $\rho < \rho_{pivot}$ then
                $r \leftarrow i$
            endif
        endif
    endfor
    if $soln = 0$ then
        no solutions exist
    endif
    convert† $A$, such that $(\mathbf{a}_j)_{rk} = 1$ and $(\mathbf{a}_j)_{ik} = 0$, $1 \leq i \leq m$, $i \neq r$
    $(\mathbf{x}_0)_r \leftarrow x_k$
endwhile
for $i = 1$ to $m$ do
    $(\mathbf{x}_0)_i^* \leftarrow (\mathbf{b}_j)_i$
endfor
for $i = m + 1$ to $n$ do
    $x_i^* \leftarrow 0$
endfor
$z^* \leftarrow e_j$
if the programme is one of minimisation then
    $z^* \leftarrow -z^*$
endif
```

Definition 90. The *two-phase method* is a procedure modified from the simplex method to cope with cases when artificial variables exist in the initial solution $\mathbf{x}_0$, in order to minimise the round-off errors that occur in the calculation. The last row in Table 3 in this case is $\mathbf{d} = \mathbf{c}^T - \mathbf{c}_0^T A = \mathbf{d}_1 + M\mathbf{d}_2$, and consequently we have Table 5 which is used here. Algorithm 3 is then firstly applied to the last row, and then again to those elements directly above the zeros in that row. When an artificial variable is removed from the first column of the table, it ceases to be basic and may be removed from the top row of the table together with the entire column under it. When the last row contains only zeros, it may be deleted from the table. The programme has no solution if non-zero artificial variables are present in the final basic set.

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Table 3 *Table used for minimisation programming using the two-phase method.*

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{d}_1$ $\mathbf{d}_2$	$-\mathbf{c}_0^T\mathbf{b}$

---

† With the use of elementary row operations.

§

Definition 91. Given a linear programme in the variables $x_1, \ldots, x_n$, there exists another linear programme associated with it, called its *dual*, which is in the variables $w_1, \ldots, w_m$. The original programme is called the *primal.* The primal completely determines the form of its dual. The *symmetric dual* of a primal linear programme in the matrix form

$$\begin{aligned} \text{minimise:} &\quad z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} &\quad A\mathbf{x} \geq \mathbf{b} \\ \text{with:} &\quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

is the linear programme

$$\begin{aligned} \text{maximise:} &\quad z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} &\quad A^T\mathbf{w} \leq \mathbf{c} \\ \text{with:} &\quad \mathbf{w} \geq \mathbf{0} \end{aligned}$$

The dual variables $w_1, \ldots, w_m$ are known as *shadow costs.* The *unsymmetric dual* of the primal

$$\begin{aligned} \text{minimise:} &\quad z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} &\quad A\mathbf{x} = \mathbf{b} \\ \text{with:} &\quad \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{maximise:} &\quad z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} &\quad A^T\mathbf{w} \leq \mathbf{c} \end{aligned}$$

The unsymmetric dual of the primal

$$\begin{aligned} \text{maximise:} &\quad z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} &\quad A\mathbf{x} = \mathbf{b} \\ \text{with:} &\quad \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{minimise:} &\quad z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} &\quad A^T\mathbf{w} \geq \mathbf{c} \end{aligned}$$

§

Note 4. We may see from Definition 91 that the dual of a programme in standard form is not itself in standard form. These duals are said to be *unsymmetric.*

§

Theorem 47. If an optimal solution exists for either the primal or the dual programme, then the other programme also has an optimal solution. If the duality is symmetric, then the two functions have the same optimal value. If the duality if unsymmetric, then the optimal value of each function can be derived from that of the other.

§

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Examples for linear programming

14th January, 2007

4. Use simplex method.

$$\begin{aligned} \text{maximise:}\quad & z = x_1 + 9x_2 + x_3 \\ \text{subject to:}\quad & x_1 + 2x_2 + 3x_3 \le 9 \\ & 3x_1 + 2x_2 + 2x_3 \le 15 \\ \text{with:}\quad & \text{all variables non-negative} \end{aligned}$$

Solution.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 1 \\ 9 \\ 1 \\ 0 \\ 0 \end{bmatrix},\ A = \begin{bmatrix} 1 & 2 & 3 & 1 & 0 \\ 3 & 2 & 2 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 9 \\ 15 \end{bmatrix},\ \mathbf{x}_0 = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix}$$

cf.

$$\begin{aligned} \text{optimise:}\quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:}\quad & A\mathbf{x} = \mathbf{b} \\ \text{with:}\quad & \mathbf{x} \ge 0 \end{aligned}$$

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\pm\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$	$\mp\mathbf{c}_0^T\mathbf{b}$

Note: $\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$ and $-\mathbf{c}_0^T\mathbf{b}$ in case of a minimisation problem, whereas $-\left(\mathbf{c}^T - \mathbf{c}_0^T A\right)$ and $\mathbf{c}_0^T\mathbf{b}$ in case of a maximisation problem.

Tableau 1;

	x_1	x_2	x_3	x_4	x_5	
x_4	1	2	3	1	0	9
x_5	3	2	2	0	1	15
	-1	-9	-1	0	0	0

The most negative number in the last row is -9. Therefore x_2-column becomes the work column. And then,

	x_2				
x_4	2	→	positive	→	$\frac{9}{2} = 4.5$
x_5	2	→	positive	→	$\frac{15}{2} = 7.5$

Since $\min(4.5, 7.5) = 4.5$, the value of x_2 on the row corresponding to x_4 becomes our pivot element. Then carry out a series of elementary row operations, namely in order $(I)_2 \leftarrow (I)_1/2$; $(II)_2 \leftarrow (II)_1 - 2(I)_2$; $(III)_2 \leftarrow (III)_1 + 9(I)_2$;

	x_1	x_2	x_3	x_4	x_5	
x_2	$\frac{1}{2}$	1	$\frac{3}{2}$	$\frac{1}{2}$	0	$\frac{9}{2}$
x_5	2	0	-1	-1	1	6
	$\frac{7}{2}$	0	$\frac{25}{2}$	$\frac{9}{2}$	0	$\frac{81}{2}$

Now the last row is all non-negative. Therefore $x_2^* = \frac{9}{2}$, $x_5^* = 6$, $x_1^* = x_3^* = x_4^* = 0$ and $z^* = \frac{81}{2}$

#

Integer programming

6^{th} December 2005

Definition 92. Algorithms which change the boundary of the solution region in order to find the optimal solution of an integer programme are called *cut algorithms*. The branch-and-bound algorithm does this by splitting the solution region into two and then discard the one which does not contain the optimal solution. The Gomory algorithm, on the other hand, reduces the feasible region with the help of a new constraint without the region being splitted.

§

Definition 93. We call *branching* a process by which a programme whose solution contains a non-integral $j < x_i < k$ is made into two separate programmes having the additional constraint $x_i \leq j$ in one, and $x_i \geq k$ in the other, the objective together with all the constraints of the original problem of which remain the same. Here j and k are positive integers and $j < k$.

§

Definition 94. In the branch-and-bound algorithm, if the objective is maximisation, the value of the objective obtained when the first integral approximation occurs is said to be the lower bound for the problem, and if the objective is minimisation it is said to be the upper bound of the same.

§

Algorithm 4 *Branch-and-bound algorithm for integer programming.*

```
find first approximation
while approximations not all integers do
    choose $x_i$ from all non-integral variables such that
        $\min\left(\left|x_i - \lfloor x_i \rfloor\right|, \left|x_i - \lceil x_i \rceil\right|\right)$ is maximised
    branch
    choose the branch whose value of the objective is maximum
endwhile
solution←last approximation
```

Example 59. (*Problem 6.9; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

Solve by branch-and-bound algorithm.

Solution. Draw a simplex table of Programme 1.

		x_1	x_2	x_3	x_4	
		1	2	1	0	
x_4	0	2	3	3	1	11
		-1	-2	-1	0	0

Replace x_4 for x_2 as the basic variable.

	x_1	x_2	x_3	x_4	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$\frac{11}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	$\frac{22}{3}$

$x_2^* = \frac{11}{3} = 3.6$, $x_1^* = x_3^* = x_4^* = 0$, $z^* = \frac{22}{3}$ Since $3 < x_2^* < 4$, branch into two programmes, namely Programme 1 where $x_2 \leq 3$, and Programme 2 where $x_2 \geq 4$. Consider first Programme 2.

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ & x_2 \leq 3 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

Use the simplex method in a tabulated form.

		x_1	x_2	x_3	x_4	x_5	
		1	2	1	0	0	
x_4	0	2	3	3	1	0	11
x_5	0	0	1	0	0	1	3
		-1	-2	-1	0	0	0

Replace the basic variable x_5 with x_2.

	x_1	x_2	x_3	x_4	x_5	
x_4	2	0	3	1	-3	2
x_2	0	1	0	0	1	3
	-1	0	-1	0	2	6

Replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1
x_2	0	1	0	0	1	3
	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	7

$x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, $z^* = 7$ Then consider Programme 3.

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ & x_2 \geq 4 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

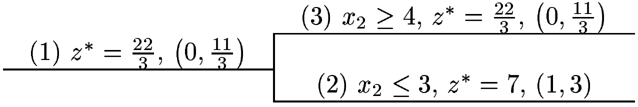
Draw a table for the two-phase method.

		x_1	x_2	x_3	x_4	x_5	x_6	
		1	2	1	0	0	$-M$	
x_4	0	2	3	3	1	0	0	11
x_6	$-M$	0	1	0	0	-1	1	4
		-1	-2	-1	0	0	0	0
		0	-1	0	0	1	-1	-15

Change x_4 for x_2 in the basic variables.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	0	0	$\frac{11}{3}$
x_6	$-\frac{2}{3}$	0	-1	$-\frac{1}{3}$	-1	1	$\frac{1}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	0	0	$\frac{22}{3}$
	$\frac{2}{3}$	0	1	$\frac{1}{3}$	1	-1	$-\frac{34}{3}$

The coefficient parts of the row corresponding to the basic variable x_6 and the last row cancel each other. The optimal result is $x_2^* = \frac{11}{3}$, $x_1^* = x_3^* = x_4^* = x_5^* = x_6^* = 0$ and $z^* = \frac{22}{3}$.


(3) $x_2 \geq 4$, $z^* = \frac{22}{3}$, $\left(0, \frac{11}{3}\right)$
(1) $z^* = \frac{22}{3}$, $\left(0, \frac{11}{3}\right)$
(2) $x_2 \leq 3$, $z^* = 7$, $(1, 3)$

Therefore the solution is $x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, and $z^* = 7$.

#

Algorithm 5 *Gomory algorithm for integer programming.*

```
while solution not wholly all integers do
    choose one non-integral optimal approximation
    write a relation from the row where that variable is basic
    rewrite the relation to make all fractional coefficients
        some integer plus a proper fraction
    move all the fractions to LHS, and all the non-fractions to RHS
    write a new constraint as LHS $\geq 0$
    find the solution for the original problem together with
        the new constraint
endwhile
```

Example 60. (*Problem 7.1; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 \\ \text{subject to:} \quad & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ \text{with:} \quad & x_1, x_2 \text{ non-negative and integral} \end{aligned}$$

Use cut algorithm.

Solve

Solution. Find the first approximation of Programme 1 normally using the simplex method.

		x_1	x_2	x_3	x_4	
		2	1	0	0	
x_3	0	2	5	1	0	17
x_4	0	3	2	0	1	10
		-2	-1	0	0	0

Since $\frac{10}{3} < \frac{17}{2}$, we know that 3 is the pivot element, and therefore we replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	$\frac{31}{3}$
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$
	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{20}{3}$

We have $x_1^* = \frac{10}{3}$, $x_3^* = \frac{31}{3}$, $x_2^* = x_4^* = 0$ and $z^* = \frac{20}{3}$. Since both x_1^* and x_3^* are non-integers, arbitrarily choose the former to generate a new constraint. Then our Programme 2 becomes,

$$\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 &= \frac{10}{3} = 3 + \frac{1}{3} \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &= 3 - x_1 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &\geq 0 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 &\geq \frac{1}{3} \\
2x_2 + x_4 &\geq 1
\end{aligned}$$

and our new programme becomes

$$\begin{aligned}
\text{maximise:}\quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 \\
\text{subject to:}\quad & \frac{11}{3}x_2 - \frac{2}{3}x_4 = \frac{31}{3} \\
& x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_4 = \frac{10}{3} \\
& 2x_2 + x_4 \geq 1 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	
		2	1	0	0	0	$-M$	
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{10}{3}$
x_3	0	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{31}{3}$
x_6	$-M$	0	2	0	1	-1	1	1
		-2	-1	0	0	0	0	0
		0	-2	0	-1	1	-1	-1

Now x_2 replaces x_6 in the basic variables and becomes the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	3
x_3	0	0	1	$-\frac{15}{6}$	$\frac{11}{6}$	$-\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	-2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	0	0	0	0

This becomes,

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	$\frac{1}{3}$	3
x_3	0	0	1	$-\frac{5}{2}$	$\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{13}{2}$

Then our first approximation of Programme 2 is $x_1^* = 3$, $x_2^* = \frac{1}{2}$, $x_3^* = \frac{17}{2}$, $x_4^* = x_5^* = 0$, and $z^* = \frac{13}{2}$. Arbitrarily choose x_2^* to generate the new constraint.

$$\begin{aligned} x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 &= \frac{1}{2} \\ \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &= -x_2 \\ \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &\geq 1 \\ x_4 - x_5 &\geq 1 \end{aligned}$$

Then our Programme 3 becomes,

$$
\begin{aligned}
\text{maximise:}\quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{subject to:}\quad & x_1 + \frac{1}{3}x_5 = 3 \\
& x_3 - \frac{5}{2}x_4 + \frac{11}{6}x_5 = \frac{17}{2} \\
& x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = \frac{1}{2} \\
& x_4 - x_5 \geq 1 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}
$$

We draw our table for this programme.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		2	1	0	0	0	0	$-M$	
x_1	0	1	0	0	0	$\frac{1}{3}$	0	0	3
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_3	0	1	0	0	$-\frac{5}{2}$	$\frac{11}{6}$	0	0	$\frac{17}{2}$
x_7	$-M$	0	0	0	1	-1	-1	1	1
		-2	-1	0	0	0	0	0	0
		0	0	0	-1	1	1	-1	-1

Then x_4 replaces the basic x_7 to become the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	1	0	0	0	$-\frac{2}{3}$	$-\frac{5}{2}$	11
x_4	0	0	0	1	-1	-1	1
	-2	-1	0	0	0	0	0

Next, x_1 remains basic and becomes a pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	-1	0	0	$\frac{1}{3}$	0	6

This becomes

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	6

The optimum point for Programme 3 is then, $x_1^* = 3$, $x_3^* = 8$, $x_4^* = 1$, $x_2^* = x_5^* = x_6^* = 0$ and $z^* = 6$. Therefore the solution to the original problem Programme 1 is $x_1^* = 3$, $x_2^* = 0$ at the objective value $z^* = 6$.

#

Definition 95. A *transportation problem* involves m *sources* each of which supplies a_i, $i = 1, \ldots, m$, units of a certain product, and n *destinations* each of which requires b_i, $i = 1, \ldots, n$, units of the same. The problem may be stated as following.

$$\begin{aligned} \text{maximise:} \quad & z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \text{subject to:} \quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \\ \text{with:} \quad & \text{all } x_{ij} \text{ non-negative and integral} \end{aligned}$$

The total supply and the total demand are assumed to be equal. Were this not so, a fictitious destination or a fictitious source is added.

§

Definition 96. The *north-west corner rule* finds an initial basic solution for the transportation algorithm of the integer programming. It begins with the (1,1) cell in the $m \times n$ table, and allocates as many units as possible to x_{11} violating neither the constraints of supply, that is the summation along each row, nor those of demand, that is the summation along each column. Then carry on moving for each step either right or downwards, until we reach the lower-right corner, x_{mn}.

§

Definition 97. A *loop*, which is a sequence of cells in the table used for finding the solution in the transportation problem, has the following properties.

a. each pair of consecutive cells is on either the same row or the same column
b. no three, or in fact any odd-numbered, consecutive cells lie in the same row or column

c. the first and the last cells are on the same row or column
d. the path along the loop is self-avoiding, that is no cells appear more than once in the sequence

§

Algorithm 6 *Transportation algorithm.*

```
while optimal solution not attained do
    find an initial, basic feasible solution using, for instance, the North-west corner rule
    let either $u_i = 0$ or $v_j = 0$ depending on whether the $i^{\mathrm{th}}$-row or the $j^{\mathrm{th}}$-column
            has the maximum number of basic solutions
    find all $u_i$ and $v_j$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$ from $u_i + v_j = c_{ij}$ for basic variables,
            and from $c_{ij} - u_i - v_j$ for non-basic variables
    improve the solution
endwhile
```

Note 5. In a transportation problem, optimal solution is achieved when $c_{ij} - u_i - u_j \geq 0$ for all transportation costs per unit c_{ij} of all non-basic variables.

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Financial mathematics

13^{th} December 2005

Definition 98. The *present value* p_0 or the *principal* is the amount initially borrowed or invested. The *future value* p_t is the principal after a period of time t.

§

Definition 99. Interest rates expressed per annum are called *nominal rates*, i. The *annual percentage rate* or *effective annual rate* i_a is the equivalent annual rate of different interest rates variously compounded.

§

Definition 100. A *sequence* is a list of numbers which follows a definite pattern. It is called an *arithmetic sequence* if each of its terms is obtained from the term immediately preceding it by an addition of a constant d, which is called the *common difference.* It is called a *geometric sequence* if each of its terms is obtained from the previous term by a multiplication of a constant r, the *common ratio.*

§

Definition 101. A *series* is the sum of the terms of sequence. It is called a *finite series* is one whose number of terms is finite, otherwise it is called an *infinite series.* An *arithmetic series* or *arithmetic progression* is the sum of the terms of an arithmetic sequence. Likewise a *geometric series* or *geometric progression* is the sum of the terms of geometric sequence.

§

Theorem 48. The value of the n^{th} term of an arithmetic series is

$$T_n = a + (n-1)d$$

The sum of its first n terms is

$$S_n = \frac{n}{2}\left(2a + (n-1)\,d\right)$$

§

Problem 41. Prove Theorem 48.

§

Theorem 49. The n^{th} term of a geometric series is

$$T_n = ar^{n-1}$$

The sum of the first n terms of it is

$$\begin{aligned} S_n &= a + ar + \cdots + ar^{n-1} \\ &= \frac{a\left(1-r^n\right)}{1-r} \\ &= \frac{a\left(r^n-1\right)}{r-1} \end{aligned}$$

When the number of terms approaches infinity, the summation in cases where $r < 1$ becomes

$$S_\infty = \frac{a}{1-r}$$

§

Definition 102. A *simple interest* is a fixed percentage of the principal paid to an investor each year. A *compound interest* is an interest paid on the principal plus any interest accumulated in previous years.

§

Theorem 50. The present value for simple interest is

$$p_t = p_0(1 + it)$$

where i is the interest rate and t the time in years.

§

Problem 42. Prove Theorem 50

§

Theorem 51. The present value in the case of compound interest is

$$p_t = p_0(1 + i)^t$$

§

Note 6. The interest may be compounded more than once a year, for example biannually, quarterly, monthly, weekly, daily, or continuously. Each time period is called a *conversion period* or *interest period.* The interest rate applied at each conversion is i/m, where m is the number of conversion periods per year. The number of conversion periods over t years is then $n = mt$.

§

Theorem 52. The present value at the end of n conversion periods is

$$p_t = p_0\left(1 + \frac{i}{m}\right)^n = p_0\left(1 + \frac{i}{m}\right)^{mt}$$

where all the variables and parameters are as previously defined.

§

Problem 43. Prove Theorem 52.

§

Theorem 53. When the number of compoundings per year becomes very large, the present value becomes

$$p_t = p_0 e^{it}$$

Proof. Since $p_t = p_0\left(1 + \frac{i}{m}\right)^{mt}$ and $\lim_{m\to\infty}\left(1 + \frac{i}{m}\right)^m = e^i$, we have the proof. ¶

Theorem 54. The annual percentage rate when compounding occurs m times per year is

$$i_a = \left(1 + \frac{i}{m}\right)^m - 1$$

§

Problem 44. Prove Theorem 54.

§

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Examples for financial mathematics

14^{th} January, 2007

5. Given the demand function for a good as $p = 1800 - 3q$. Find the coefficient of point elasticity of demand when p is 300, 900 and 1200. Describe in words each of the results. Find the percentage change if the price of good increases by 10 per cent.

Solution. The coefficient of point elasticity of demand is $\varepsilon_d = -\frac{1}{b}\frac{p_0}{q_0}$, and also $\varepsilon_d = \frac{\Delta q_d}{\Delta p}$, where Δq_d is percentage change in quantity demanded and Δp percentage change in price.

	p_0	300	900	1200
q_0	$\frac{(1800-p_0)}{3}$	500	300	200
ε_d	$-\frac{1}{3}\frac{p_0}{q_0}$	$-\frac{1}{5}$	-1	-2
$\lvert\varepsilon_d\rvert$	$\frac{1}{3}\frac{p_0}{q_0}$	< 1	1	> 1
demand		elastic	unit elastic	inelastic
Δq_d	$\varepsilon_d \Delta p$	-2%	-10%	-20%

#

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Integral calculus

10^{th} January 2006

Definition 103. Let $f(x)$ be a function, and let $f'(x)$ be its derivative. The reverse process of differentiation is called *antidifferentiation* or *integration.* It gives us the original function, which is called the *antiderivative* or *integral* of $f(x)$.

§

Theorem 55. Let c, n and k be constants. Then

a.

$$\int k\,dx = kx + c$$

b.

$$\int dx = x + c$$

c.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1$$

d.

$$\int x^{-1}\,dx = \ln x + c, \quad x > 0$$

e.

$$\int x^{-1}\,dx = \ln|x| + c, \quad 0 \neq x < 0$$

f.

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$

g.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

h.

$$\int (f(x) \pm g(x)) = \int f(x)\,dx \pm \int g(x)\,dx$$

i.

$$\int -f(x)\,dx = -\int f(x)\,dx$$

§

Definition 104. The approximation $\sum_{i=1}^{n}(f(x_i)\,\Delta x^i)$ of the area under a continuous curve A is called a *Riemann sum.* That area under the curve is

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i$$

§

Theorem 56. Let $F(x)$ be the integral of $f(x)$. We call the *fundamental theorem of calculus* the expression.

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

§

Theorem 57.

a.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

b.

$$\int_a^a f(x)dx = F(a) - F(a) = 0$$

c.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, \quad a \le b \le c$$

d.

$$\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b (f(x) \pm g(x))dx$$

e.

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

§

Property d of Theorem 57 is used to find the area between two curves.

Theorem 58. The process of *integration by parts* is

$$\int (f(x) \cdot g'(x))dx = f(x) \cdot g(x) - \int (g(x) \cdot f'(x))dx$$

Proof. From

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

we have

$$f(x) \cdot g(x) = \int (f(x) \cdot g'(x))dx + \int (g(x) \cdot f'(x))dx$$

¶

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Examples for integral calculus

14^{th} January, 2007

6. Find $\int_0^2 x^2\,dx$, $\int_{-2}^2 \left(4 - x^2\right) dx$ and $\int_{-1}^1 \frac{1}{x^3}\,dx$.

Solution.

$$\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

#

$$\int_{-2}^2 \left(4 - x^2\right) dx = \left(4x - \left.\frac{x^3}{3}\right|_{-2}^2 = \frac{32}{3}$$

#

Consider $\int_{-1}^1 \frac{1}{x^3}\,dx = F(1) - F(-1)$. Here $F(1) = 1$ and $F(-1) = -1$ But $\infty = F(0^+) \neq F(0^-) = -\infty$, which means that $f(x) = 1/x^3$ is not continuous at 0, which is a point in $[-1, 1]$. Therefore the definite integral given does not exist.

#

7. Find $\int \frac{2z}{\sqrt[3]{z^2}+1}\,dz$

Solution. (1) We know that if u is a differentiable function of x, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

provided that $n \neq -1$. Let $u = z^2 + 1$, then $du = 2z dz$. And then,

$$\begin{aligned}\int \frac{2z}{\sqrt[3]{z^2}+1} &= \int \frac{1}{u^{\frac{1}{3}}}\,du \\ &= \int u^{-\frac{1}{3}}\,du \\ &= \frac{3}{2}u^{\frac{2}{3}} + C \\ &= \frac{3}{2}\left(z^2+1\right)^{\frac{2}{3}} + C\end{aligned}$$

#

Solution. (2) Let $u = \sqrt[3]{z^2} + 1$, then $3u^2 du = 2z dz$.

$$\begin{aligned}
\int \frac{2z}{\sqrt[3]{z^2}+1} dz &= \int \frac{3u^2}{u} du \\
&= 3 \int u du \\
&= 3\frac{u^2}{2} + C \\
&= \frac{3}{2}\left(z^2+1\right)^{\frac{3}{2}} + C
\end{aligned}$$

#

8. Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx$.

Solution. (1) Let $u = x^3 + 1$. Then $du = 3x^2 dx$, $u(-1) = 0$ and $u(1) = 2$. And then,

$$\begin{aligned}
\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx &= \int_{0}^{2} \sqrt{u}du \\
&= \left.\frac{2}{3}u^{\frac{3}{2}}\right|_{0}^{2} \\
&= \frac{4\sqrt{2}}{3}
\end{aligned}$$

Solution. (2) Again, substitute the u and du as above. Then find the indefinite integral,

$$\begin{aligned}
\int 3x^2\sqrt{x^2+1}dx &= \int \sqrt{u}du \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}} + C
\end{aligned}$$

And then consider the corresponding definite integral,

$$\begin{aligned}
\int_{-1}^{1} 3x^2\sqrt{x^3+1}dx &= \left.\frac{2}{3}\left(x^3+1\right)^{\frac{3}{2}}\right|_{-1}^{1} \\
&= \frac{4\sqrt{2}}{3}
\end{aligned}$$

#

9. Find the average value of $f(x) = \sqrt{4-x^2}dx$ on the interval $[-2, 2]$.

Solution. The average value required is,

$$\frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}dx = \frac{\pi}{2}$$

#

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Exercises for Integral calculus

14^{th} January, 2007

10. Find the definite integrals,

$\int\left(20x^6+3x^4-6x^3\right)dx$, $\int\frac{1}{x+1}dx$, $\int 8\sqrt{x-7}dx$,

$\int 4e^{-3.5t}dt$, and $\int 3x^{-\frac{2}{3}}dx$.

11. Find the definite integral

$$\int\left(2e^{3t}-3e^{-5t}\right)dt$$

given an initial, or a boundary condition $F(0)=3$.

12. Find the values of the definite integrals, $\int_1^3 5x^3dx$, $\int_1^2 4e^{\frac{t}{2}}dt$, $\int_2^5 6x^{-3}$, and $\int_{-3}^{-1}(-4)e^{-2t}dt$.

13. Find a firm's total revenue r_t function, given the marginal revenue function $r_m=-.2x^2-1.3x+500$.

14. Let the present value be $p=a^{-rt}$ of the sum of money a to be received in the future when the interest is compounded continuously. Find the present value p_n of a stream of future income, that is to say, the money to be received each year for n years.

15. Find the following integrals and definite integrals.

$$\begin{array}{lll}
\int 8x^{\frac{2}{3}}dx & \int_1^2 8x^{\frac{2}{3}}dx & \int\sqrt{x+2}dx \\
\int_1^2\sqrt{x+2}dx & \int\frac{3}{t^4}dt & \int_1^2\frac{3}{t^4}dt \\
\int\left(10-2x^2\right)dx & \int_1^2\left(10-2x^2\right)dx & \int 4(x-3)^{-3}dx \\
\int_1^2 4(x-3)^{-3}dx & \int 10e^{1.2t}dt & \int_1^2 10e^{1.2t}dt \\
\int\frac{3x^4}{\sqrt{3x^2-3}}dx & \int_1^2\frac{3x^4}{\sqrt{3x^2-3}}dx & \int\frac{1}{\sqrt{x}}dx \\
\int_1^2\frac{1}{\sqrt{x}}dx & \int\left(\frac{1}{\sqrt{x}}+\frac{1}{x}+x\right)dx & \int_1^2\left(\frac{1}{\sqrt{x}}+\frac{1}{x}+x\right)dx \\
\int x(6x+3)dx & \int_1^2 x(6x+3)dx & \int\left(3x^3+9x^2\right)dx \\
\int_1^2\left(3x^3+9x^2\right)dx & \int\left(x^{\frac{3}{4}}+\frac{1}{x}\right)dx & \int_1^2\left(x^{\frac{3}{4}}+\frac{1}{x}\right)dx
\end{array}$$

$\int (5+x)^2 dx$
$\int_1^2 (5+x)^2 dx$
$\int \left(3\alpha^{\frac{3}{8}} + \frac{1}{2x} + e^x\right) dx$
$\int_1^2 \left(3\alpha^{\frac{3}{8}} + \frac{1}{2x} + e^x\right) dx$
$\int 2x(x+3)^2 dx$
$\int_1^2 2x(x+3)^2 dx$
$\int \left(1 - e^{-3t}\right) dt$
$\int_1^2 \left(1 - e^{-3t}\right) dt$
$\int \left(e^5 + e^{-2t} + \frac{1}{t}\right) dt$
$\int_1^2 \left(e^5 + e^{-2t} + \frac{1}{t}\right) dt$
$\int \frac{1}{x} dx$
$\int_1^2 \frac{1}{x} dx$
$\int \left(e^{2t} + 3t^2 + \frac{1}{t}\right) dt$
$\int_1^2 \left(e^{2t} + 3t^2 + \frac{1}{t}\right) dt$
$\int e^3 dt$
$\int_1^2 e^3 dt$
$3\int u^{\frac{1}{2}} du$
$3\int_1^2 u^{\frac{1}{2}} du$
$\int -5 dx$
$\int_1^2 -5 dx$
$\int \frac{1}{x^5} dx$
$\int_1^2 \frac{1}{x^5} dx$
$\int xe^x dx$
$\int_1^2 xe^x dx$
$\int 5e^{-2t} dt$
$\int_1^2 5e^{-2t} dt$
$\int \frac{x}{x+1} dx$
$\int_1^2 \frac{x}{x+1} dx$
$\int \left(6\sqrt{x} + \frac{3}{x}\right) dx$
$\int_1^2 \left(6\sqrt{x} + \frac{3}{x}\right) dx$
$\int e^{\frac{t}{3}} dt$
$\int_1^2 e^{\frac{t}{3}} dt$
$\int \left(5x^{\frac{1}{4}} + \frac{1}{x} + e^x\right) dx$
$\int_1^2 \left(5x^{\frac{1}{4}} + \frac{1}{x} + e^x\right) dx$
$\int 3e^{2t} dt$
$\int_1^2 3e^{2t} dt$
$\int \left(2\sqrt{x} + \frac{1}{x}\right) dt$
$\int_1^2 \left(2\sqrt{x} + \frac{1}{x}\right) dt$
$\int x\sqrt{x+7} dx$
$\int_1^2 x\sqrt{x+7} dx$
$\int 3\sqrt{x} dx$
$\int_1^2 3\sqrt{x} dx$
$\int \left(12\sqrt{x} - \frac{1}{x}\right)$
$\int_1^2 \left(12\sqrt{x} - \frac{1}{x}\right) dx$
$\int -\sqrt{x} dx$
$\int_1^2 -\sqrt{x} dx$
$\int \left(\frac{2}{x} + e^x\right) dx$
$\int_1^2 \left(\frac{2}{x} + e^x\right) dx$
$\int \left(\frac{1}{x} + 2e^x\right) dx$
$\int_1^2 \left(\frac{1}{x} + 2e^x\right) dx$
$\int 3x^{-1} dx$
$\int_1^2 3x^{-1} dx$
$\int \frac{2x}{(x+1)^2} dx$
$\int_1^2 \frac{2x}{(x+1)^2} dx$
$\int 2(x-1)^{-2} dx$
$\int_1^2 2(x-1)^{-2} dx$

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Integral calculus

7^{th} February 2006

Definition 105.

a. $\int_a^a f(x)dx = 0$
b. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

§

Theorem 59.

a. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$. And if $k=-1$, then

$$\int_a^b -f(x)dx = -\int_a^b f(x)dx$$

b. $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
c. If $f(x) \geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ Let $g(x)=0$. Then, $f(x) \geq 0$ on $[a,b]$ implies $\int_a^b f(x)dx \geq 0$
d. If $\max f$ and $\min f$ are the maximum and minimum values of f on $[a,b]$, then

$$\min f \cdot (b-a) \leq \int_a^b f(x)dx \leq \max f \cdot (b-a)$$

e. If f if integrable on the intervals between a, b and c, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

§

Theorem 60 is called *60.*

Theorem 60. If f is continuous on the closed innterval $[a,b]$, then at some point c in the interval $[a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

§

Theorem 61. The *average-* or *mean value* of an integrable function f on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

§

Theorem 62. If f has a constant value c on $[a,b]$, then

$$\int_a^b f(x)dx = \int_a^b cdx = c(b-a)$$

§

Theorem 63. If f is continuous on $[a, b]$, then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point on $[a, b]$ and

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

§

Theorem 64. If f is continuous at every point of $[a, b]$ and F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

§

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Simultaneous equations

24^{th} January 2005

Definition 106. We call *system of equations* equations which together describe a mathematical model. A system of equations of n equations and v variables is called an $n \times v$ system or a system with $n \times v$ dimensions. If $n = v$ the system of equations is called an *exactly constrained system*, if $n < v$ an *under-constrained system*, and if $n > v$ an *over-constrained system*.

§

Note 7. A *unique solution* to a system exists only if there are as many equations as variables, that is to say, if $n \geq v$. An under-constrained system may have an unlimited number of solutions or no solutions, but it may never have a unique solution. An exactly constrained or over-constrained system may have a unique solution, an infinite number of solutions, or no solution.

§

Definition 107. The graph of a (2×2) linear system of equations comprise two straight lines. If the two lines intersect, then the point of intersection (x_1, y_1) satisfies both equations and therefore represents a unique solution of the system. If they do not intersect, then there are no solutions and the two corresponding equations are said to be *inconsistent* with each other. If the two equations have identical graph, then the system has an infinite number of solutions. Such equations are called *dependent* or *equivalent* equations.

§

Note 8. Consider a (2×2) system of linear equations in the slope-intercept form,

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

```
if $m_1 \neq m_2$ then
  system has a unique solution
else
  if $b_1 \neq b_2$ then
    equations are inconsistent and the system has no solution
  else
    equations are equivalent and the system has infinitely many solutions
```

§

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Differential equation

31^{st} January 2005

Definition 108. A *differential equation* (DE) is an equation which involves derivatives. An *ordinary differential equation* (ODE) is a differential equation in which there is exactly one independent variable. A *partial differential equation* (PDE) is one where there are at least two independent variables. The derivatives of an ODE are ordinary-, whereas those of a PDE are partial derivatives.

§

Definition 109. Consider a differential equation. The *order* of it is the order of the highest derivative appearing in it. Its *degree* is the degree of the highest ordered derivative therein. A *primitive* is a relation between the variables that involves n essential arbitrary constants, which gives rise to a differential equation of order n. The n constants are called *essential* if they cannot be replaced by a smaller number of constants.

§

Example 61. The differential equation $y''' + 3\left(y''\right)^2 + 2y' = \sin x$ is an ordinary differential equation of order 3 and degree one. The differential equation $\left(y''\right)^2 + \left(y'\right)^3 + y = 2x$ is an ODE which has an order 2 and degree 2.

Problem 45. The problem of finding solutions of differential equations is essentially that of finding the primitive which gave rise to the equation.

§

Example 62. The differential equation $y''' = 0$ has a primitive $y = Ax^2 + Bx + C$, $y''' - 6y'' + 11y' - 6y = 0$ has $y = C_1e^{3x} + C_2e^{2x} + C_3e^x$, $y^2\left(y''\right)^2 + y^2 = r^2$ has $(x - C)^2 + y^2 = r^2$.

§

Definition 110. *Existence theorems* give conditions by which one could determine whether a differential equation is solvable. A *particular solution* of a differential equation is one obtained from the primitive by assigning definite values to the parameters, that is to say, the arbitrary constants. A *singular solution* is a solution which cannot be obtained from the primitive by any manipulation of the arbitrary constants. The primitive of a differential equation is usually called the *general solution* of the equation.

§

Definition 111. A differential equation is said to be *variable separable* if an integrating factor can be readily found. Such equation has the form

$$f_2(x) \cdot g_2(y)dx + f_2(x) \cdot g_1(y)dy = 0$$

Through the use of the integrating factor

$$\frac{1}{f_2(x) \cdot g_2(y)}$$

the primitive of this is then

$$\int \frac{f_1(x)}{f_2(x)} dx + \int \frac{g_1(y)}{g_2(y)} dy = C$$

§

Definition 112. A differential equation of the first order and first degree may be written in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

If this such equation admits a solution $f(x,y,C) = 0$ where C is an arbitrary constant, then there exist infinitely many integrating factors $xi(x,y)$ such that $\xi(x,y)\,[M(x,y)\,dx + N(x,y)\,dy] = 0$ is exact, and there exist transformations of the variables which render the latter separated. But since no general rules exist for doing this, the use in practice is still somewhat limited.

§

Definition 113. A function $f(x,y)$ is said to be *homogeneous* of degree n if

$$f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

§

Note 9. The equation

$$(a_1 x + b_1 y + c_1)\, dx + (a_2 x + b_2 y + c_2)\, dy = 0$$

where $a_1 b_2 - a_2 b_1 = 0$, is reduced through the transformation

$$a_1 x + b_1 y = t \quad \text{and} \quad dy = \frac{dt - a_1\, dx}{b_1}$$

to the form

$$P(x,t)\,dx + Q(x,t)\,dt = 0$$

§

Note 10. The equation

$$(a_1 x + b_1 y + c_1)\, dx + (a_2 x + b_2 y + c_2)\, dy = 0$$

where $a_1 b_2 - a_2 b_1 \neq 0$, is reduced through the transformation

$$x = x' + h \quad \text{and} \quad y = y' + k$$

in which $x = h$ and $y = k$ are the solutions of the equations $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$ into the homogeneous form

$$(a_1 x' + b_1 y')\, dx' + (a_2 x' + b_2 y')\, dy' = 0$$

§

Note 11. The equation of the form

$$y \cdot f(xy)\,dx + x \cdot g(xy)\,dy = 0$$

through the transformation

$$xy = z, \qquad y = \frac{z}{x}, \qquad dy = \frac{x\,dz - z\,dx}{x^2}$$

into the form

$$P(x,y)\,dx + Q(x,z)\,dz = 0$$

which is variable separable.

§

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Examples for differential equations

14^{th} January, 2007

16. Solve $x^2(y+1)\,dx + y^2(x-1)\,dy = 0$.

Solution. The integrating factor is $\frac{1}{(y+1)(x-1)}$ by the multiplicative application of which the given equation becomes

$$\frac{x^2}{x-1}dx + \frac{y^2}{y+1}dy = 0$$

And then from this,

$$\begin{aligned}
\left(x+1+\frac{1}{x-1}\right)dx + \left(y-1+\frac{1}{y+1}\right)dy &= 0 \\
\int\left(x+1+\frac{1}{x-1}\right)dx + \int\left(y-1+\frac{1}{y+1}\right)dy &= 0 \\
\frac{x^2}{2} + x + \ln(x-1) + \frac{y^2}{2} - y + \ln(y+1) &= C_1 \\
x^2 + 2x - 2y + \ln(x-1)(y+1) &= C_2 \\
(x+1)^2 + (y-1)^2 + 2\ln(x-1)(y+1) &= C
\end{aligned}$$

#

Difference equation

20^{th} February 2005

Definition 114. A *difference equation* gives the relationship between an *independent variable* and a *dependent* variable, which changes at fixed, equally spaced intervals in time. The *order* of a difference equation is the number of time intervals spanned within the equation.

§

Example 63. In the difference equation $y_{t+1} = 1.1y_t$, an independent variable is the time t while the dependent variable is the income y. The order is the span of t intervals within the equation, which in this case is $(t+1)-t = 1$.

Theorem 65. The general solution of a homogeneous first-order difference equation is of the form $y_t = Aa^t$, where t and y are respectively the independent and dependent variables.

§

Note 12. Given a homogeneous first-order difference equation of the form $y_{t+1} - by_t = 0$ and some conditions, we may find the parameters a and A of our general solution $y_t = Aa^t$ by first finding a by substitution of this general solution for t and $t+1$ in the difference equation, and then find A from the conditions given.

§

Theorem 66. The stability of the solution to a difference equation in general form $y_t = Aa^t$ is,

range of a	time path of y_t	*solution*	*time path*
$-\infty < a < -1$	$a^t \to \pm\infty$	unstable	alternates
$-1 < a < 0$	$a^t \to 0$	stable	alternates
$0 < a < 1$	$a^t \to 0$	stable	tends to zero
$1 < a < \infty$	$a^t \to \infty$	unstable	tends to infinity

§

Theorem 67. The solution of a non-homogeneous difference equation is the sum of a complementary function and a particular integral, that is $y_t = y_c + y_p$. The *complementary function* is the solution of the homogeneous part of the difference equation. The *particular integral* is a function which satisfies the full difference equation.

§

Note 13. The general form of the particular integral is deduced from the right-hand side of the difference equation. In particular, if c and b are constants,

right-hand side	*general form of particular integral*
c	$y_p = k$
cb^t	$y_p = kb^t$

§

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Teresa Bradley and Paul Patton. *Essential mathematics for economics and business.* 2nd ed. 2002(1998)

Examples for difference equations

14^{th} January, 2007

17. Solve the difference equation $y_{t+1} - 1.1y_t = 0$ by iteration for year 2, 3, 4 and 5, given the income in year 1 is 20,000 Bahts..

Solution. We have $y_{t+1} = 1.1y_t$, therefore

$$\begin{aligned} y_1 &= 20000 \\ y_2 &= 1.1(20000) = 22000 \\ y_3 &= 1.1(22000) = 24200 \\ y_4 &= 1.1(24200) = 26620 \\ y_5 &= 1.1(26620) = 29282 \end{aligned}$$

#

18. Write out the solution of the difference equation $y_{t+1} - 1.1y_t = 0$ for $t = 1, 2, 3, 4$ and 5 in terms of y_1. Deduce the general expression for y_t in terms of y_1. Evaluate y_{40} given $y_t = 20000$.

Solution.

$$\begin{aligned} y_2 &= 1.1y_1 \\ y_3 &= 1.1y_2 = 1.1\,(1.1y_1) = (1.1)^2 y_1 \\ y_4 &= 1.1y_3 = 1.1(1.1)^2 y_1 = (1.1)^3 y_1 \\ y_5 &= 1.1y_4 = 1.1(1.1)^3 y_1 = (1.1)^4 y_1 \end{aligned}$$

#

In general,

$$y_t = (1.1)^{t-1} y_2$$

#

$t = 40$, $y_1 = 20000$;

$$y_{40} = (1.1)^{39} 20000 = 822895.56$$

#

19. Find a general solution of the difference equation $y_{t+1} - 0.9y_t = 0$. If $y_2 = 100$, find the particular solution. Evaluate y_1, y_2, y_3, y_{20} and y_{50}.

Solution. The general form of the solution is $y_t = Aa^t$. Therefore $y_{t+1} = Aa^{t+1}$. Then the difference equation becomes

$$\begin{aligned} y_{t+1} - 0.9y_t &= 0 \\ Aa^{t+1} - 0.9a^t &= 0 \\ Aa^t(a - 0.9) &= 0 \end{aligned}$$

Since $A \neq 0$ and $a^t \neq 0$, therefore $a - 0.9 = 0$, that is $a = 0.9$. Thus the general solution is

$$y_t = A(0.9)^t$$

#

From $y_2 = 100$;

$$\begin{aligned} y_2 &= A(0.9)^2 \\ 100 &= A(0.9)^2 \\ A &= 123.46 \end{aligned}$$

The particular solution is then

$$y_p = 123.46(0.9)^t$$

#

Then we tabulate the required calculation,

t	y_t
1	$123.46(0.9) = 111.11$
2	$123.46(0.9)^2 = 100.00$
3	$123.46(0.9)^3 = 90.00$
20	$123.46(0.9)^{20} = 15.01$
50	$123.46(0.9)^{50} = 0.64$

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Teresa Bradley and Paul Patton. *Essential mathematics for economics and business.* 2nd ed. 2002(1998)

Exercises for difference equation

14^{th} January, 2007

20. For each of the following difference equations, state

i The order of the equation.

ii Whether the equation is homogeneous or not.

(a) $p_{t+1} - 0.8p_t = 0$ (b) $y_{t+2} = 8 - y_{t+1}$ (c) $y_{t+2} = 80 + y_t$

21. Solve each of the following difference equations for the indicated variable by the iteration method.

a. $y_{t+1} - 0.8y_t = 10$, given $y_1 = 1$. Then find y_5.

b. $p_{t+2} = 4p_{t+1} - 8p_t$, given $p_1 = 20$, $p_2 = 18$. Then find p_5.

c. $p_t = 0.6p_{t-1} + 80$, given $p_1 = 100$. Find p_5.

22. An amount of money A is invested at $r\%$ compounded annually.

a. Show that the value of the investment at the end of any year t is given by the difference equation $p_{t+1} = (1 + r)p_t$

b. Show that the general solution of the equation is $p_t = p_0(1 + r)^t$.

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Teresa Bradley and Paul Patton. *Essential mathematics for economics and business.* 2^{nd} ed. 2002(1998)

Function

A *function of one independent variable* is a relation in the form $y = \mathrm{f}(x)$ such that there exists one and only one value of y in the range of f for each real number x in the domain of f. The variable y is called the *dependent variable.*

An *implicit function* is a function in which both dependent- and independent variables appear on the same side.

An *explicit function* is one where the dependent variable is on the left hand side-, and the independent variable on the right hand side of the equation.

Polynomial

$$a_n x^n + \ldots + a_1 x + a_0$$

Polynomial equation

$$a_n x^n + \ldots + a_1 x + a_0 = 0$$

Quadratic equation

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Definition of limits

If f(x) is a function which draws closer to a unique finite real number l for all values of x as the latter draws closer to a, but $x \neq a$, then l is called the *limit* of f(x) as x approaches a. In notation this is,

$$\lim_{n \to a} \mathrm{f}(x) = l$$

Definition of limits

For a function $\mathrm{f}(x)$, $\lim_{n\to a}\mathrm{f}(x) = l$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathrm{f}(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

Rules of limits

$$\lim_{x\to a} k = k$$

$$\lim_{x\to a} x^n = a^n$$

$$\lim_{x\to a} k\mathrm{f}(x) = k\lim_{x\to a}\mathrm{f}(x)$$

$$\lim_{x\to a}[\mathrm{f}(x) \pm \mathrm{g}(x)] = \lim_{x\to a}\mathrm{f}(x) \pm \lim_{x\to a}\mathrm{g}(x)$$

$$\lim_{x\to a}[\mathrm{f}(x)\cdot\mathrm{g}(x)] = \lim_{x\to a}\mathrm{f}(x)\cdot\lim_{x\to a}\mathrm{g}(x)$$

$$\lim_{x\to a}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \frac{\lim_{x\to a}\mathrm{f}(x)}{\lim_{x\to a}\mathrm{g}(x)}, \text{ provided that } \lim_{x\to a}\mathrm{g}(x)\neq 0$$

$$\lim_{x\to a}[\mathrm{f}(x)]^n = \left[\lim_{x\to a}\mathrm{f}(x)\right]^n, \text{ provided that } n>0$$

Definition of a derivative

Let $y = \mathrm{f}(x)$. Then, the derivative of y with respect to x is,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0}\frac{\mathrm{f}(x+h)-\mathrm{f}(x)}{h}$$

The various notations for the derivative include

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x},\ \frac{\mathrm{d}f}{\mathrm{d}x},\ f'(x),\ y',\ \mathrm{D}y \,\text{and}\, \mathrm{D}(\mathrm{f}(x))$$

General rules of differentiation

Let u, v and w are functions of x, and c is a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cx) = c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cx^n) = ncx^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u \pm v \pm \ldots) = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x} \pm \ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(cu) = c\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uvw) = uv\frac{\mathrm{d}w}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) - u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)}{v^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(u^n) = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}u}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}u}}{\frac{\mathrm{d}x}{\mathrm{d}u}}$$

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

$$\frac{\mathrm{d}\ln x}{\mathrm{d}x} = \frac{1}{x}$$

Laws of exponents

Let p and q be real numbers, a and b positive numbers, and m and n positive integers. Then,

$$a^p \cdot a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$(a^p)^q = a^{pq}$$

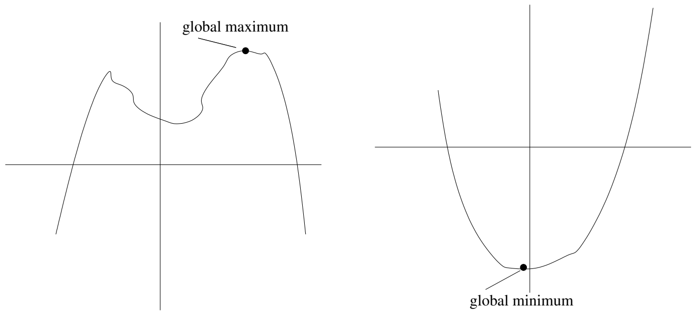
$$a^0 = 1, \text{ provided that } a \neq 0$$

$$a^{-p} = \frac{1}{a^p}$$
$$(ab)^p = a^p b^p$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Laws of logarithms

$$\log_a mn = \log_a m + \log_a n$$
$$\log_a \frac{m}{n} = \log_a m - \log_a n$$
$$\log_a m^p = p \log_a m$$
$$\log_a n = \frac{\log_b n}{\log_b a}$$

Kit Tyabandha, PhD Department of Mathematics, Mahidol University



(a) (b)

Figure 1 *(a) global maximum-, and (b) global minimum points of a function.*

Business mathematics, Graph and derivative, 25th October 2005 –15– From 21st October 2005 , as of 14th January, 2007

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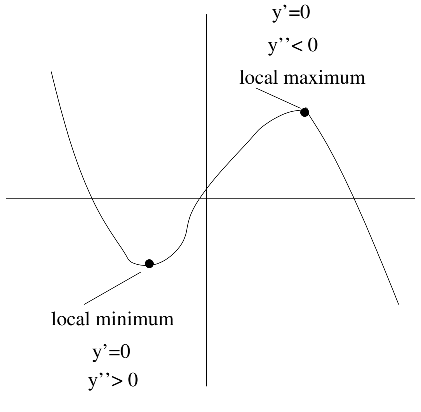

Figure 2 *Local minimum- and local maximum points of a function.*

Business mathematics, Graph and derivative, 25th October 2005 –16– From 21st October 2005 , as of 14th January, 2007

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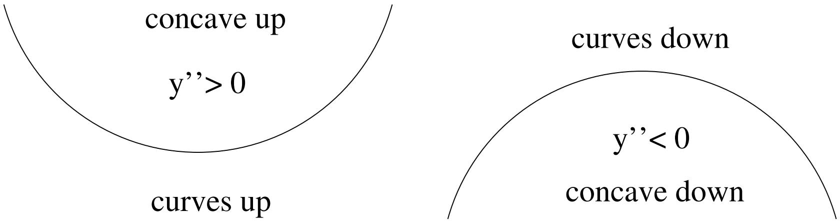

(a) (b)

Figure 3 *The two curvature types, namely (a) concave up ($y'' > 0$) and, (b) concave down ($y'' < 0$).*

Business mathematics, Graph and derivative, 25th October 2005 –17– From 21st October 2005 , as of 14th January, 2007

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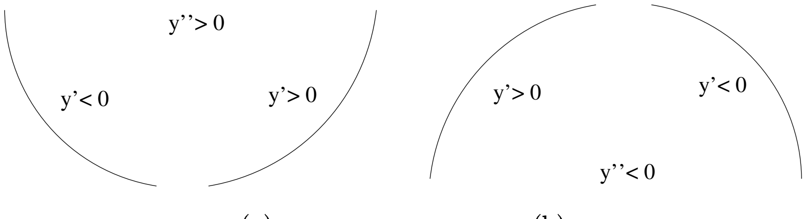
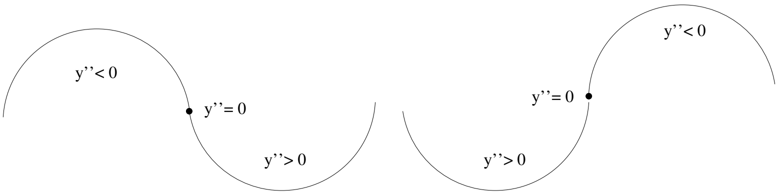


(a) (b)

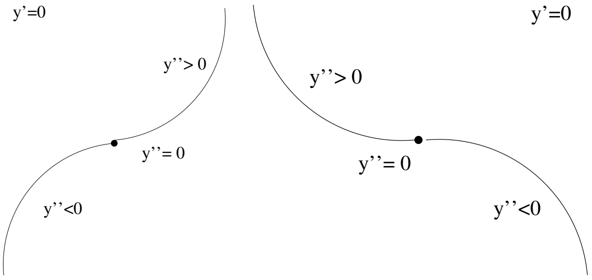
Figure 4 *The four possible curvatures in two dimensions, considering both y' and y'', (a) $y'' > 0$ and, (b) $y'' < 0$.*

Business mathematics, Graph and derivative, 25th October 2005 –18– From 21st October 2005 , as of 14th January, 2007



(a) (b)

Figure 5 *Inflection points, where $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*



(a) (b)

Figure 6 *Stationary inflection points, where both $y' = 0$ and $y'' = 0$, (a) with y'' increasing and, (b) with y'' decreasing.*

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Figure 7 shows a plot of the case where $a = 1$, $b = 2$, $c = 3$ and $d = 4$.

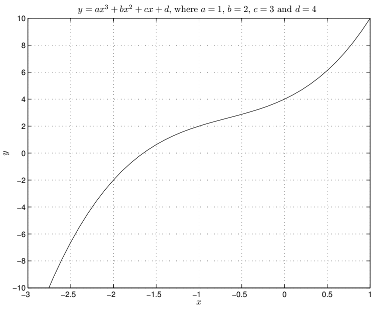

Figure 7 *The cubic function* $y = x^3 + 2x^2 + 3x + 4$.

Business mathematics, Graph and derivative, 25th October 2005 –21– From 21st October 2005 , as of 14th January, 2007

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Figure 8 shows the case where $a = 1$, $b = 5$, $c = 4$ and $d = 3$

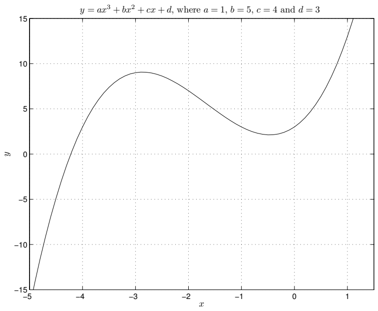


Figure 8 *The cubic function* $y = x^3 + 5x^2 + 4x + 3$.

Business mathematics, Graph and derivative, 25th October 2005 –22– From 21st October 2005 , as of 14th January, 2007

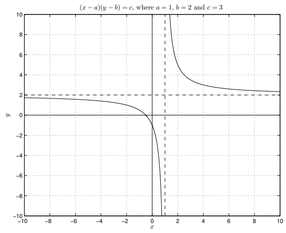


Figure 9 *The hyperbolic function* $(x-1)(y-2)=3$.

The *general demand function* is of the form

$$q_d = \mathrm{f}(p, y, p_s, p_c, t_a, a, \ldots)$$

where q_d is the quantity demand of good x, p the price of x, y the income of the consumer, p_s the price of substitute goods, p_c the price of complementary goods, t_a the taste or fashion of the consumer, and a the advertisement level.

In its simplest case where all other factors are constant, the *demand equation* takes the form

$$p = c_1 - c_2 q_d$$

where p is the price-, while q_d the quantity demanded of the good x, and c_1 and c_2 are positive constants.

The *general supply function* is of the form

$$q_s = \mathrm{f}(p, c, p_0, t_e, n, o, \ldots)$$

where q_s is the quantity supplied of good x, p the price of x, c the cost of production, p_0 the price of other goods, t_e the available technology, n the number of producers in the market, and o other factors, for example tax and subsidies.

The simplified relation for the *supply* is

$$p = c_1 + c_2 q_s$$

where q_s is the quantity of x supplied, and c_1 and c_2 are positive constants.

Definition 15 *Total cost and revenue*
The *total cost* c_t comprises a fixed cost and a variable cost, that is

$$c_t = c_f + c_v$$

The *total revenue* r_t is price times output,

$$r_t = pq$$

Definition 16 *Marginal cost and revenue*

The *marginal cost* c_m is the change in total cost caused by the production of an additional unit.

The *marginal revenue* r_m is the change in total revenue coming from the sale of an extra good. In other words,

$$c_m = \frac{\mathrm{d}c_t}{\mathrm{d}q} \text{ and}$$

$$r_m = \frac{\mathrm{d}r_t}{\mathrm{d}q}$$

where q is the output.

Definition 17 *Average cost and revenue*

The *average cost* c_a is the total cost per unit output,

$$c_a = \frac{c_t}{q}$$

The *average revenue* r_a is the total revenue per unit output,

$$r_a = \frac{r_t}{q}$$

From Definition 15, $r_t = pq$, and from Definition 17, $r_t = r_a q$, therefore

$$p - r_a$$

From Definition 17, $c_a = \frac{c_t}{q}$, and from Definition 15, $c_a = c_f + c_v$. Therefore

$$c_a = c_{af} + c_{av}$$

where the average fixed cost $c_{af} = \frac{c_f}{q}$ and the average variable cost $c_{av} = \frac{c_v}{q}$.

Procedure 1 *Total-, average-, and marginal graphs*

Given: $c_t(x)$
$c_a \leftarrow \frac{c_t}{x}$
$c_m \leftarrow c_t'$

```
**for** each $f \in \{c_t, c_a, c_m\}$ **do**
    **find** the critical values $\mathbf{x}_c$ for $f' = 0$
    **find** $f''$
    $n \leftarrow |\mathbf{x}_c|$
    **for** $i = 1$ to $n$ **do**
        **if** $f''(x_i^c) > 0$ **then**
            $f(x_i^c)$ is convex and is the relative minimum of $f$
        **elseif** $f''(x_i^c) < 0$ **then**
            $f(x_i^c)$ is concave and is the relative maximum of $f$
        **endif**
    **endfor**
    **find** inflection points $\mathbf{x}_f$ from $f'' = 0$
**endfor**
**plot** $c_t(x)$, then $c_a(x)$ and $c_m(x)$
```

The relationship between input and output is called a *production function*,

$$q = \mathrm{f}(l, k, r, t_e, s, e, \ldots)$$

where l is labour, k phical capital such as buildings and machines, r raw materials, t_e technology, s land, and e enterprise.

Assuming a short period of time, then l becomes the only independent variable and the other remaining factors are parameters, that is fixed, and therefore $q = \mathrm{f}(l)$.

Then the *marginal product of labour* is

$$p_{lm} = \frac{\mathrm{d}q}{\mathrm{d}l}$$

and the *average product of labour* is

$$p_{la} = \frac{q}{l}$$

The *marginal propensity to consume* is $p_{cm} = \frac{\mathrm{d}c}{\mathrm{d}y}$.

The *marginal propensity to save* is $p_{sm} = \frac{\mathrm{d}s}{\mathrm{d}y}$.

The *average propensity to consume* is $p_{ca} = \frac{c}{y}$.

The *average propensity to save* is $p_{sa} = \frac{s}{y}$.

Here y is the income, c the consumption, and s the saving.

The *profit* is

$$\pi = r_t - c_t$$

At the *break-even point*

$$\pi = 0$$

that is

$$r_t = c_t$$

Calculus of multivariable functions

Definition 21 *functions of n independent variables*

A function $y = \mathrm{f}(x_1, \ldots, x_n)$ is called a *function of n independent variables* if there exists one and only one value of y in the range of f for each tuple of real number $(x_1, \ldots, x_n)$ in the domain of f. Here y is called the *dependent variable* while x_i, $i = 1, \ldots, n$, the *independent variables.*

Definition 22 *partial derivatives*

Let a multivariable function be

$$y = \mathrm{f}(x_1, \ldots, x_n)$$

The *partial derivative* of y with respect to x_i, where $1 \le i \le n$, is a measure of the instantaneous rate of change of y with respect to x_i while x_j is held constant for all $j \neq i$, where $1 \le j \le n$. This partial derivative is defined as

$$\frac{\partial y}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{\mathrm{f}(\ldots, x_i + \Delta x_i, \ldots) - \mathrm{f}(x_1, \ldots, x_n)}{\Delta x_i}$$

and can be written in either one of the following forms.

$$\frac{\partial y}{\partial x_i}, \ \frac{\partial \mathrm{f}}{\partial x_i}, \ \mathrm{f}_{x_i}(x_1, \ldots, x_n), \ \mathrm{f}_{x_i}, \text{ or } y_{x_i}$$

Theorem 9 *product rule*
Let $z = \mathrm{g}(x, y) \cdot \mathrm{h}(x, y)$. Then,

$$\frac{\partial z}{\partial x} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial x} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = \mathrm{g} \cdot \frac{\partial \mathrm{h}}{\partial y} + \mathrm{h} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

Theorem 11 *generalised power function rule*
Let $z = [\mathrm{g}(x, y)]^n$. Then,

$$\frac{\partial z}{\partial x} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial x}$$

and

$$\frac{\partial z}{\partial y} = n\mathrm{g}^{n-1} \cdot \frac{\partial \mathrm{g}}{\partial y}$$

Definition 23 *second-order partial derivatives*
Let $z = \mathrm{f}(x, y)$. Then, the *second-order direct partial derivatives* are

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

These are also written

$$\mathrm{f}_{xx},\ (\mathrm{f}_x)_x,\ \frac{\partial^2 z}{\partial x^2} \quad \text{and respectively} \quad \mathrm{f}_{yy},\ (\mathrm{f}_y)_y,\ \frac{\partial^2 z}{\partial y^2}$$

The *cross partial derivatives* are

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

These are also written as

$$\mathrm{f}_{xy},\ (\mathrm{f}_x)_y,\ \frac{\partial^2 z}{\partial y \partial x} \quad \text{and respectively} \quad \mathrm{f}_{yx},\ (\mathrm{f}_y)_x,\ \frac{\partial^2 z}{\partial x \partial y}$$

Theorem 12 *critical points*

For a multivariable function $z = \mathrm{f}(x, y)$ to be a *relative maximum* at (a, b) necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} < 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ at that point. For the same at the same to be a *relative minimum*, necessarily $\mathrm{f}_x, \mathrm{f}_y = 0$, and $\mathrm{f}_{xx}, \mathrm{f}_{yy} > 0$ and $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ there. Moreover, an *inflection point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, and both f_{xx} and f_{yy} have the same sign. On the other hand, a *saddle point* is a point (a, b) at which $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$, but f_{xx} and f_{yy} are of different signs.

Procedure 2 *Procedure for determining a critical point of a function with two independent variables*

```
Given $z = \mathrm{f}(x, y)$ and a point $(a, b)$, at this point,
if $\mathrm{f}_x = 0$ and $\mathrm{f}_y = 0$ then
  $(a, b)$ is a critical point
  if $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ then
    if $\mathrm{f}_{xx} < 0$ and $\mathrm{f}_{yy} < 0$ then
      $(a, b)$ is a relative maximum of $z$
    elseif $\mathrm{f}_{xx} > 0$ and $\mathrm{f}_{yy} > 0$ then
      $(a, b)$ is a relative minimum of $z$
    else †
    endif
    elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < (\mathrm{f}_{xy})^2$ then
        if $\mathrm{f}_{xx} \cdot \mathrm{f}yy > 0$ then
            $(a, b)$ is an inflection point
        elseif $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} < 0$ then
            $(a, b)$ is a saddle point
        else ‡
        endif
    else
        test inconclusive
    endif
else
    $(a, b)$ is no critical point
endif
```

Problem 15 *details in the critical point procedure*

There are two dead ends in Procedure 2. The first one (†) is the case where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} > (\mathrm{f}_{xy})^2$ and either $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} < 0)$, $(\mathrm{f}_{xx} = 0, \mathrm{f}_{yy} > 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} = 0)$, $(\mathrm{f}_{xx} < 0, \mathrm{f}_{yy} > 0)$, or $(\mathrm{f}_{xx} > 0, \mathrm{f}_{yy} < 0)$. The second one (‡) is where $\mathrm{f}_{xx} \cdot \mathrm{f}_{yy} = 0$. Find out what happen in these cases, and thus complete the missing lines of logic in Procedure 2.

Definition 24 *derivative and differential*

By *derivative* $\frac{\mathrm{d}y}{\mathrm{d}x}$ we mean an infinitesimally small change in y with respect to an infinitesimally small change in x. By *differential* $\mathrm{d}y$ and $\mathrm{d}x$ we mean an infinitesimally small change in the values of y and respectively x.

Example 13 *derivative and differential of functions of one variable and two variables*

For a function of one variable $y = \mathrm{f}(x)$, the total derivative is

$$\frac{\mathrm{d}y}{\mathrm{d}x}$$

and the differential of y is

$$\mathrm{d}y = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{d}x$$

For a function of two variables $z = \mathrm{f}(x, y)$ partial derivatives are, the first-order partial derivatives

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}$$

and the second-order partial derivatives

$$\frac{\partial^2 z}{\partial x^2} \equiv z_{xx},\ \frac{\partial^2 z}{\partial y^2} \equiv z_{yy},\ \frac{\partial^2}{\partial y \partial x} \equiv z_{xy} \text{ and } \frac{\partial^2 z}{\partial x \partial y} \equiv z_{yx}$$

The total differential of z is

$$\mathrm{d}z = \left(\frac{\partial \mathrm{f}}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial \mathrm{f}}{\partial y}\right) \mathrm{d}y$$

and for small changes which are not infinitesimal, dx becomes Δx and the incremental change formula is

$$\Delta z \approx \left(\frac{\partial \mathrm{f}}{\partial x}\right) \Delta x + \left(\frac{\partial \mathrm{f}}{\partial y}\right) \Delta y$$

Definition 25 *general production function*

The *general production function* is $q = \mathrm{f}(l, k)$, where q is output of the production, l labour and k capital. The *Cobb-Douglas production function* in its general form is

$$q = al^{\alpha}k^{\beta} \tag{1}$$

where a is a constant and $0 < \alpha < 1$, $0 < \beta < 1$, $l > 0$ and $k > 0$.

Example 14 *Cobb-Douglas production function*

With the Cobb-Douglas production function, the *marginal product of labour* is,

$$p_{lm} = q_l = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{2}$$

and the *marginal product of capital*

$$p_{km} = q_k = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{3}$$

From this we see that $p_{lm} > 0$ and $p_{km} > 0$.

Theorem 13 *law of diminishing returns to labour*

From the Cobb-Douglas production function we have the *law of diminishing returns to labour*, which states that $q_{ll} < 0$.

Proof. From Equation 1 in Definition 25,

$$q_{ll} = \frac{\partial^2 q}{\partial l^2} = \frac{\partial}{\partial l}\left(\frac{\partial q}{\partial l}\right) = \frac{\partial p_{lm}}{\partial l} = (\alpha - 1)\frac{\alpha q}{l^2}$$

Since $0 < \alpha < 1$, therefore $q_{ll} < 0$.

¶

Example 15 *changes in marginal product values*
Using the Cobb-Douglas production function,

$$q_{kl} = q_{lk} = a\alpha\beta l^{\alpha-1}k^{\beta-1}$$

Therefore, $q_{lk} > 0$ and $q_{kl} > 0$. In other words, p_{lm} increases as capital input k increases, and respectively p_{km} increases as labour input l increases.

Example 16 *average functions of labour and capital*

For the Cobb-Douglas production function in Equation 1 the average product of labour is

$$p_{la} = \frac{q}{l} = al^{\alpha-1}k^{\beta} \tag{4}$$

and the average product of capital is

$$p_{ka} = \frac{q}{k} = al^{\alpha}k^{\beta-1} \tag{5}$$

Example 17 *marginal functions of labour and capital*

Again using the Cobb-Douglas production function of Equation 1, the marginal product of labour is

$$p_{lm} = \frac{\partial q}{\partial l} = a\alpha l^{\alpha-1}k^{\beta} \tag{6}$$

and the marginal product of capital is

$$p_{km} = \frac{\partial q}{\partial k} = a\beta l^{\alpha}k^{\beta-1} \tag{7}$$

Example 18 *comparison between marginal and average functions*

From the APL equation, Equation 4, and the MPL equation, Equation 6, and since $0 < \alpha < 1$, therefore $p_{ml} < p_{la}$. Similarly from the APK equation, Equation 5, and the MPK equation, Equation 7, since $0 < \beta < 1$, we have $p_{km} < p_{ka}$.

Example 19 *conditions for using labour and capital*

A producer likes to have a positive marginal function, which means that the productivity increases as the input increases. But the second derivative is negative, which means that this rate of increase slows down as time goes by. In practice, the conditions for using labour are,

$$p_{lm} = \frac{\partial q}{\partial l} > 0,\ \frac{\mathrm{d}p_{lm}}{\mathrm{d}l} = \frac{\partial^2 q}{\partial l^2} < 0, \text{ and} \, p_{lm} < p_{la} \tag{8}$$

The conditions for using capital are similarly,

$$p_{km} = \frac{\partial q}{\partial k} > 0,\ \frac{\mathrm{d}p_{km}}{\mathrm{d}k} = \frac{\partial^2 q}{\partial k^2} < 0, \text{ and} \, p_{km} < p_{ka} \tag{9}$$

Definition 26 *production function graphs*

An *isoquant* is a graph in two dimensions, $k = \mathrm{f}(l)$, plotted to represent a production function $q = \mathrm{f}(l, k)$. The slope

$$\frac{\mathrm{d}k}{\mathrm{d}l}$$

is called the *marginal rate of technical substitution*. The value of this slope at (l_0, k_0) is denoted by

$$\left.\frac{\mathrm{d}k}{\mathrm{d}l}\right|_{l_0 k_0}$$

Theorem 15 *slope of an isoquant*

The slope of an isoquant is the ratio of the marginal products.

Proof. The total differential of $q = \mathrm{f}(l, k)$ is

$$\mathrm{d}q = \left(\frac{\partial q}{\partial l}\right)\mathrm{d}l + \left(\frac{\partial q}{\partial k}\right)\mathrm{d}k$$

Along any isoquant, $\mathrm{d}q = 0$, therefore,

$$0 = \left(\frac{\partial q}{\partial l}\right)\mathrm{d}l + \left(\frac{\partial q}{\partial k}\right)\mathrm{d}k \tag{10}$$

This directly yield, after some manipulation,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{q_l}{q_k}$$

Or, from Equation 10 together with Equation's 6 and 7, it follows that,

$$\frac{\mathrm{d}k}{\mathrm{d}l} = -\frac{p_{lm}}{p_{km}}$$

Definition 27 *returns to scale*

In the Cobb-Douglas production function equation, Equation 1, let both inputs l and k change by the same proportion, and let λ be the constant of this proportionality. Then $q_2 = a(\lambda l)^\alpha(\lambda k)^\beta$, which leads to $q_2 = \lambda^{\alpha+\beta}q_1$. When $\alpha + \beta = 1$, the case is described as *constant returns to scale*, when $\alpha + \beta < 1$ as *decreasing returns to scale*, and when $\alpha + \beta > 1$ as *increasing returns to scale.*

Definition 28 *homogeneous Cobb-Douglas production function*

The *homogeneous* Cobb-Douglas production function of order r is,

$$\mathrm{f}(\lambda l, \lambda k) = \lambda^r \mathrm{f}(l, k)$$

where $r = (\alpha, \beta)$.

Definition 29 *utility function*
A *utility function* expresses utility as a function of goods consumed. In its general form this is,

$$u = \mathrm{f}(x, y)$$

where x and y are the quantities of goods X and respectively Y consumed.

Definition 30 *Cobb-Douglas utility function*
The *Cobb-Douglas utility function* is in its general form,

$$u = ax^{\alpha}y^{\beta}$$

where a is a constant, and $0 < \alpha < 1$, $0 < \beta < 1$, $x > 0$ and $y > 0$.

Definition 31 *marginal utility*

The *marginal utility* for a utility function with one variable, $u = \mathrm{f}(x)$, is $\frac{\mathrm{d}u}{\mathrm{d}x} = u_x = u_{xm}$. The marginal utility for a utility function with two variables, $u = \mathrm{f}(x, y)$, is $\frac{\partial u}{\partial x} = u_x = u_{xm}$ and $\frac{\partial u}{\partial y} = u_y = u_{ym}$.

Definition 32 *indifference curves*

The *indifference curve* is a graph $y = \mathrm{f}(x)$ drawn to represent a utility function $u = \mathrm{f}(x, y)$. Its slope $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called the *marginal rate of substitution.* Setting the total differential equal to zero,

$$0 = \mathrm{d}u = \left(\frac{\partial u}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial u}{\partial y}\right) \mathrm{d}y$$

we find

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_x}{u_y}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{u_{xm}}{u_{ym}}$$

Definition 33 *partial elasticities of demand*

Let a demand function be

$$q_a = \mathrm{f}(p_a, y, p_b) \tag{11}$$

where q_a is the quantity demanded of good a, p_a the price of a, y consumer's income, and p_b the price of another good b. Then, the *price elasticity of demand* is,

$$\varepsilon_d = \frac{\partial q_a}{\partial p_a}\frac{p_a}{q_a}$$

The *income elasticity of demand* is,

$$\varepsilon_y = \frac{\partial q_a}{\partial y}\frac{y}{q_a}$$

And the *cross-price elasticity of demand* is,

$$\varepsilon_c = \frac{\partial q_a}{\partial p_b}\frac{p_b}{q_a}$$

Example 20 *partial elasticity with respect to labour*
With the demand function as in Equation 11, the partial elasticity with respect to labour is,

$$\varepsilon_{ql} = \frac{\partial q}{\partial l}\frac{l}{q}$$

And from Equation's 6 and 4, this leads to,

$$\varepsilon_{ql} = \frac{p_{lm}}{p_{la}}$$

For the Cobb-Douglas production function, Equation 1, then $\varepsilon_{ql} = \alpha$.

Example 21 *partial elasticity with respect to capital*
Again, with the demand function as in Equation 11, the partial elasticity with respect to capital is,

$$\varepsilon_{qk} = \frac{\partial q}{\partial k}\frac{k}{q}$$

Then, from Equation's 7 and 5,

$$\varepsilon_{qk} = \frac{p_{km}}{p_{ka}}$$

For the Cobb-Douglas production function, Equation 1, we have $\varepsilon_{qk} = \beta$.

Definition 34 *function*

A *function* is an operator or a procedure which accepts a permissible input and transforms it into a unique output. The input is some nonempty set. If a function is defined to be

$$y = \mathrm{f}(\mathrm{x})$$

then x is the input vector, y the output, and f(·) the function itself.

Example 22 *function*

If f(·) is the function of dressing, then its input is possibly a person and its output a dressed person.

If f(·) is the function of making up, then the input is perhaps a girl and the output a made-up girl.

Definition 35 *variables and parameters of a function*

Let

$$y = \mathrm{f}(\mathrm{a}, \mathrm{x})$$

be a function, where a is a set of all its parameters, and x a set of all its variables. Then y is its dependent variable and x_i, for all $i \in \mathrm{x}$, are its independent variables.

In other words, x_i vary, y follows, and a_i could assume any value within the range of its permissible ones, but its value must be constant.

Definition 36 *inverse function*

An *inverse function* is an expression of the independent variable in terms of the dependent variables. The inverse of the function $\mathrm{f}(\cdot)$ is denoted by $\mathrm{f}^{-1}(\cdot)$. If $\mathrm{f}(\cdot)$ is a function which admits one independent variable, namely x, then one could express it as,

$$y = \mathrm{f}(x) \tag{12}$$

Its inverse function is then,

$$\mathrm{f}^{-1}(y) = x \tag{13}$$

Example 23 *operator and inverse operator*

Both the function and its inverse may be thought of as being an operator operating on an input to produce an output. The function,

$$y = \mathrm{f}(x)$$

is understood diagrammatically as,

$$y \leftarrow \boxed{\ \mathrm{f}(\cdot)\ } \leftarrow x$$

while its inverse function,

$$x = \mathrm{f}^{-1}(y)$$

is displayed as a diagram as,

$$y \rightarrow \boxed{\ \mathrm{f}^{-1}(\cdot)\ } \rightarrow x$$

Theorem 16 *inverse function*

An inverse function must always be a one-to-one mapping.

Proof. Let $f(\cdot)$ be a function. Then $f(\cdot)$ can be either one-to-one or many-to-one, and therefore $f^{-1}(\cdot)$ could turn out to be either one-to-one or one-to-many. But since f^{-1} is also a function, so for each of the values in its domain the corresponding value in its range must be unique. This means that in cases where f^{-1} turns out to be one-to-many, some constraints must be put on its input in order to make the output one-to-one, which then makes all the outputs from $f^{-1}(\cdot)$ one-to-one. ¶

$f(\cdot)$	$f^{-1}(\cdot)$
addition	subtraction
multiplication	division
power	root
exponential	logarithm

Table 1 *Some of the functions and their corresponding inverse functions.*

$f(\cdot)$	$f^{-1}(\cdot)$
$x + a$	$x - a$
$x \cdot a$	$\frac{x}{a}$
x^a	$\sqrt[a]{x}$
a^x	$\log_a x$

Table 2 *The notational forms of functions and their inverses.*

in which division and logarithm are both undefined for $a = 0$.

Definition 37 *some inverse functions*

The inverse of the addition,

$$y = x + a$$

is the subtraction,

$$y - a = x$$

The inverse of the multiplication,

$$y = ax$$

is the division,

$$\frac{y}{a} = x$$

The inverse of the power,

$$y = x^a$$

is the root,

$$\sqrt[a]{y} = x$$

The inverse of the exponential,

$$y = a^x$$

is the logarithm,

$$\log_a y = x$$

Example 25 *building-blocks of mathematics*

Figure 10 *addition makes multiplication makes power function*

$$\begin{tikzcd}
 & \underbrace{x\cdots x}_{b} \arrow[r] & x^b \\
 & x\cdot x = x^2 \arrow[u] & \\
\underbrace{x+\ldots+x}_{a} \arrow[r] & x\cdot a \arrow[u] & \\
x+x=2x \arrow[u] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \underbrace{a\cdots a}_{x} \arrow[r] & a^x \\
\underbrace{a+\ldots+a}_{n} \arrow[r] & a\cdot n \arrow[u] & \\
a+a=2a \arrow[u] & &
\end{tikzcd}$$

Figure 11 *Starting from a constant to obtain in the end the exponential function.*

Definition 38 *exponential function*

An *exponential function* is defined as

$$y = a^x$$

where $a > 0$ and $a \neq 1$.

Example 26 *exponential function*

The domain of the exponential function $y = a^x$ is the set of all real numbers, while its range the set of all positive real numbers. The function is convex and increasing when $a > 1$, and convex and decreasing when $0 < a < 1$. At $x = 0$, the value of the function is $y = 1$ for any $a > 0$.

Theorem 17 *exponential to the power of zero*

For any $a \neq 0$,

$$\lim_{x \to 0} a^x = 1$$

Theorem 18 *rules of exponential function*
Three basic rules of the exponential function are,

$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(a^m)^n = a^{mn}$$

Proof. Write, say, a^m as,

$$\underbrace{a \cdots a}_{m}$$

and similarly for a^n. Then all three equations above become obvious. ¶

Example 27 *graphs of the exponential function*

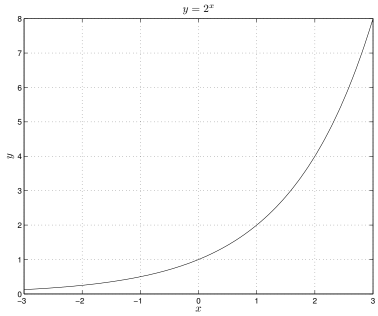


Figure 12 *Example of the graph of the exponential function when* $a > 1$*. Here the graph is that of* $y = 2^x$*.*

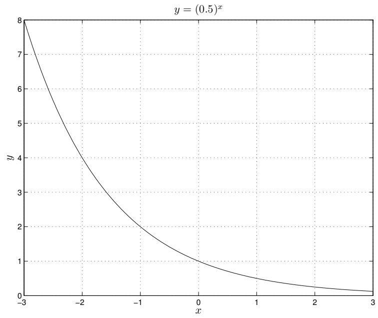


Figure 13 *An example of graph of the exponential function* $y = a^x$ *when* $0 < a < 1$*. Here* $a = 0.5$*.*

Definition 39 *growth- and decay curves*
Let $a > 1$. Then the graph of

$$y = a^x$$

is called a *growth curve*, while that of

$$y = a^{-x}$$

is called a *decay curve*.

Example 28 *growth functions*
There are basically three laws of growth, namely unlimited, limited and logistic growth, all of which involve an exponential function. The model is for unlimited growth,

$$y(t) = ae^{rt}$$

for limited growth,

$$y(t) = m\left(1 - e^{-rt}\right)$$

and for logistic growth,

$$y(t) = \frac{m}{1 + ae^{-rmt}}$$

where a, m and r are constants.

Example 29 *interest compounding*

The value of a principal p compounded annually at an interest rate i for t years is,

$$s = p(1+i)^t$$

where i is expressed in decimal points. For compounding m times a year, then,

$$s = p\left(1+\frac{i}{m}\right)^{mt}$$

If the compounding is continuous, at 100 per cent interest for one year, then,

$$s = p\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = pe$$

where e is the Euler's constant, $e = 2.71828\ldots$

Example 30 *multiple compounding*

For multiple compounding,

$$p(1+i_e)^t = p\left(1+\frac{i}{m}\right)^{mt}$$

the effective annual rate of interest is,

$$i_e = \left(1+\frac{i}{m}\right)^{m} - 1$$

The effective annual rate of interest for continuous compounding is,

$$i_e = e^r - 1$$

Definition 40 *discounting*

Discounting is the process of finding the present value p of a future sum of money s.

Example 31 *discounting*

Discounting when under annual compounding is,

$$s = p(1+i)^t$$

when under multiple compounding,

$$p = s\left(1+\frac{i}{m}\right)^{-mt}$$

and when under continuous compounding,

$$p = se^{-rt}$$

When discounting, the interest rate i is called the *rate of discount.*

Example 32 *converting exponential- to natural exponential functions*

A discrete growth $s = p(1 + i/m)^{mt}$ can be converted to a continuous growth $s = pe^{rt}$ thus,

$$\begin{aligned} p\left(1+\frac{i}{m}\right)^{mt} &= pe^{rt} \\ \ln\left(1+\frac{i}{m}\right)^{mt} &= \ln e^{rt} \\ r &= m \ln\left(1+\frac{i}{m}\right) \end{aligned}$$

Therefore,

$$s = p\left(1+\frac{i}{m}\right)^{mt} = pe^{m \ln\left(1+\frac{i}{m}\right)t}$$

Example 33 *reflections of graphs*

Reversing the sign of x, that is replacing x by $-x$, has the effect of reflection of the original graph with respect to the y-axis. Reversing the sign of y, that is replacing y by $-y$, gives a reflection of the same with respect to the x-axis. The graphs of $y = a^{\pm x}$ remain always above the x-axis, in other words the function $y = a^{\pm x}$ maps $-\infty < x < \infty$ to $y > 0$. The two functions $y = a^x$ and $y = a^{-x}$ are the reflection of each other with respect to the y-axis. It can be easily seen that the functions $y = -a^{\pm x}$ are the reflection with respect to the x-axis respectively of $y = a^{\pm x}$.

Definition 41 *logarithmic function*

The *logarithmic function* with base a is defined to be the inverse of the exponential function, and is written

$$y = \log_a x$$

where $a > 0$ and $a \neq 1$. The logarithmic function of base 10 is called the *common logarithmic function*, and one of base e, where

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

is called the *natural logarithmic function.* By the notation $y = \log_a x$ we mean that the logarithm base a of x is the power to which a must be raised to get x.

Example 34 *logarithmic function*

The domain of the logarithmic function

$$y = \log_a x$$

is the set of all positive real numbers, its range the set of all real numbers.

The function is concave and increasing for $a > 1$, and is convex and decreasing for $0 < a < 1$. Note also that $\log_a x$ is the power which a must be raised to get x.

Example 35 *examples of natural logarithm*

Note that

$$e^{\ln a} = a = \ln e^{a}$$

where $a > 0$,

$$e^{\ln x} = x = \ln e^{x}$$

where $x > 0$, and

$$e^{\ln \mathrm{f}(x)} = \mathrm{f}(x) = \ln e^{\mathrm{f}(x)}$$

where $\mathrm{f}(x) > 0$.

Theorem 19 *rules of logarithm*

Four basic rules for logarithm function are listed in the following.

$$\log_b m + \log_b n = \log_b mn$$

$$\log_b m - \log_b n = \log_b \frac{m}{n}$$

$$\log_b m^{z} = z \log_b m$$

$$\log_b n = \frac{\log_x n}{\log_x b}$$

Definition 42 *elasticity of substitution*
The *elasticity of substitution* σ is defined as,

$$\sigma = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\frac{k}{l}}}{\frac{\mathrm{d}\left(\frac{p_l}{p_k}\right)}{\frac{p_l}{p_k}}} = \frac{\frac{\mathrm{d}\left(\frac{k}{l}\right)}{\mathrm{d}\left(\frac{p_l}{p_k}\right)}}{\frac{\frac{k}{l}}{\frac{p_l}{p_k}}}$$

where $\frac{k}{l}$ is called the *least-cost input ratio*, and $\frac{p_l}{p_k}$ the *input-price ratio.*

Example 36 *values of the elasticity of substitution*
The value $\sigma = 0$ means there is no substitutability, that is the two inputs are complements of each other and both must be used together in a fixed proportion.

The value $\sigma = \infty$ means that the two goods may substitute each other perfectly. Ultimately, $0 \leq \sigma \leq \infty$.

Definition 43 *constant elasticity of substitution production function*
A *constant elasticity of substitution production function* is a production function where, unlike the Cobb-Douglas function, has an elasticity of substitution whose value is constant but not necessarily 1. In its typical form, it is,

$$q = a\left(\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right)^{-\frac{1}{\beta}}$$

where a is called the *efficiency parameter*, α the *distribution parameter*, β the *substitution parameter*. Furthermore, β determines σ, and $a > 0$, $0 < \alpha < 1$, and $\beta > -1$.

Example 37 *logarithmic transformation of nonlinear functions*
Some nonlinear functions can be converted to linear functions using logarithmic transformation, for example the Cobb-Douglas production function,

$$q = ak^{\alpha}l^{\beta}$$

which becomes

$$\ln q = \ln a + \alpha \ln k + \beta \ln l$$

Other nonlinear functions can not be converted, for example the constant elasticity of substitution production function,

$$q = a\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]^{-\frac{1}{\beta}}$$

which becomes just another nonlinear function,

$$\ln q = \ln a - \frac{1}{\beta}\ln\left[\alpha k^{-\beta} + (1-\alpha)l^{-\beta}\right]$$

others

Example 38 *nonlinear total revenue*

Let the total revenue be

$$r_t = pq$$

and the demand function

$$p = a - bq$$

where q is the quantity sold. Then r_t expressed as a function of q is nonlinear, for

$$r_t = (a - bq)q = aq - bq^2$$

Example 39 *nonlinear total cost*

A more realistic equation for the total cost instead of

$$c_t = a + bq$$

is the nonlinear function

$$c_t = aq^3 - bq^2 + cq + d$$

in which the production cost increases with quantity in at a decreasing rate ($c_t' < 0$) up to the inflection point at

$$q = \frac{b}{6a}$$

after which it increases at an increasing rate ($c_t'' > 0$).

Definition 44 *polynomial*

A *polynomial* is an expression in the form

$$\sum_{i=0}^{n} a_i x^{n-i}$$

Here n is called the *order* of the polynomial. If $n = 2$ the polynomial is known as a *quadratic polynomial*, if $n = 3$ a *cubic polynomial*, if $n = 4$ a *quartic*, $n = 5$ a *quintic* and $n = 6$ a *sextic*. If we let $\mathrm{p}(x)$ be a polynomial, then a *polynomial equation* is the equation $\mathrm{p}(x) = 0$. A *polynomial function* is a function of the form

$$y = \mathrm{f}(x) = \mathrm{p}(x)$$

Example 40 *quadratic equation*

The quadratic equation $ax^2 + bx + c = 0$ has the solutions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{14}$$

These solutions are called the *roots* of the quadratic equation. Equation 14 is called the 'minus-b formula'. The values of x obtained from the minus-b formula give the intersections of the graph of the quadratic function

$$\mathrm{f}(x) = \mathrm{p}(x) = ax^2 + bx + c$$

on the x-axis.

The value $b^2 - 4ac$ determines how the graph of $\mathrm{f}(x)$ lies relative to the x-axis, that is,

$$b^2 - 4ac \begin{cases} > 0, & \text{there are two } x\text{-intersections} \\ = 0, & \text{the graph touches the } x\text{-axis at one point} \\ < 0, & \text{the graph never touches the } x\text{-axis} \end{cases}$$

Furthermore, the graph reverses its direction with respect to the y-axis at the critical point where $\mathrm{f}'(x) = 0$, that is when

$$x = -\frac{b}{2a}$$

Consequently the critical point is

$$\left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right)$$

The graph of f(x) is symmetric with respect to the vertical line which passes through the turning point, that is to say, the line

$$x = -\frac{b}{2a}$$

The y-intercept is at the point $(0, c)$.

Example 41 *hyperbolic function*

A hyperbolic relation has the form,

$$(px + q)(ry + s) = t$$

From this we obtain,

$$\begin{aligned}\left(x + \frac{q}{p}\right)\left(y + \frac{s}{r}\right) &= \frac{t}{pr} \\ y &= \frac{t}{pr}\left(\frac{1}{x + \frac{q}{p}}\right) - \frac{s}{r} \\ &= \frac{a}{x + b} - c\end{aligned}$$

where p, q, r, s and t are constants, hence so are $a = \frac{t}{pr}$, $b = \frac{q}{p}$, $c = \frac{s}{r}$.

In economics we sometimes find hyperbolic functions of the form,

$$y = \frac{a}{bx + c} \tag{15}$$

For example, a demand function of a good may be given by,

$$q + a = \frac{m}{p}$$

which leads to,

$$p = \frac{m}{q + a}$$

where p and q are respectively price and quantity demanded of a good, while m and a are constants.

The graph of Equation 15 has the x-axis, that is the line $y = 0$, as its horizontal asymptote, and has the line

$$x = -\frac{c}{b}$$

as its vertical asymptote. If all the parameters are positive, then the curve in the first quadrant decreases with a decreasing rate.

Definition 45 *matrix addition*
Let $A = \{a_{ij}\}$, $B = \{b_{ij}\}$ and $C = \{c_{ij}\}$ be three matrices. Then

$$C = A + B$$

is called the *addition* of the matrices A and B if

$$c_{ij} = a_{ij} + b_{ij}$$

for all i and j.

Definition 46 *matrix multiplication*
Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and $\mathbf{B} = (b_{kl})$ an $n \times p$ matrix. Then the product $\mathbf{AB}$ is an $m \times p$ matrix $\mathbf{C} = (c_{il})$ where,

$$c_{il} = \sum_{k=1}^{n} a_{ik} b_{kl}$$

where $1 \leq i \leq m$ and $1 \leq l \leq p$.

Definition 47 *determinant*

The expression obtained by eliminating the n variables $x_1, \ldots, x_n$ from n equations,

$$\left.\begin{array}{r} a_{11}x_1 + \ldots + a_{1n}x_n = 0 \\ \vdots\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n = 0 \end{array}\right\} \tag{16}$$

is called the *determinant* of this system of equations, Equation 16. The determinant of matrix A denoted by various different notations, for example $\det(A)$, $|A|$, $\sum(\pm a_1 b_2 c_3 \cdots)$, $D(a_1 b_2 c_3 \cdots)$, or $|a_1 b_2 c_3 \cdots|$.

Example 42 *determinant of systems of three variables*

For a linear system of three variables, Equation 16 can be written as,

$$\left.\begin{array}{r} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \\ c_1x + c_2y + c_3z = 0 \end{array}\right\} \tag{17}$$

Eliminating x, y and z from Equation 17 gives us,

$$a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1 = 0$$

Definition 48 *minor*

A *minor* M_{ij} of any matrix A is the determinant of a reduced matrix obtained by omitting the i^{th} row and the j^{th} column of A.

Theorem 20 *Laplacian expansion*

Determinant can be determined by,

$$|\,A\,| = \sum_{i=1}^{k} a_{ij} C_{ij}$$

where C_{ij} is called the *cofactor* of a_{ij}. The cofactor C_{ij} can also be denoted as a^{ij}, and,

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is a minor of A.

Definition 49 *permutation inversion*
Any pairwisely ordered pair in a permutation p is called a *permutation inversion* in p if $i > j$ and $p_i < p_j$.

Theorem 21 *determination of determinant by permutation*
Determination of the determinant can also be determined by,

$$|\, A \,| = \sum_{\pi} (-1)^{\mathrm{I}(\pi)} \prod_{i=1}^{n} a_{i,\pi(i)}$$

where π is a permutation which ranges over all permutations of $\{1, \ldots, n\}$, and $\mathrm{I}(\pi)$ is called the *inversion number* of π.

Theorem 22 *properties of determinant*

Let a be a constant and A an $n \times n$ matrix. Then,

$$|\,aA\,| = a^n\,|\,A\,|$$

$$|\,-A\,| = (-1)^n\,|\,A\,|$$

$$|\,AB\,| = |\,A\,|\,|\,B\,|$$

$$|\,I\,| = |\,AA^{-1}\,| = |\,A\,|\,|\,A^{-1}\,| = 1$$

$$|\,A\,| = \frac{1}{|\,A^{-1}\,|}$$

Definition 50 *multilinearity*

A function in two or more variables is said to be *multilinear* if it is linear in each variable separately.

Theorem 23 *multilinearity of determinants*
Determinants of matrix are multilinear in rows and columns.

Example 43 *multilinearity of determinants*
Consider an 3 × 3 matrix,

$$A = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

What Theorem 23 says about multilinearity of determinants is the same as saying that,

$$|A| = \begin{vmatrix} a_1 & 0 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & 0 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

and

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} + \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

Definition 51 *conformal mapping*

A *conformal mapping* is a transformation that preserves local angle. The terms *function*, *map* and *transformation* are synonyms.

Definition 52 *similarity transformation*

A *similarity transformation* is a conformal mapping the transformation matrix of which is,

$$A' \equiv BAB^{-1}$$

Here A and A' are similar matrices.

Theorem 24 *similarity transformation and determinant*
Similarity transformation does not change the determinant.

Proof. The proof for this is simply,

$$|\,BAB^{-1}\,| = |\,B\,|\,|\,A\,|\,|\,B^{-1}\,| = |\,B\,|\,|\,A\,|\,\frac{1}{|\,B\,|} = |\,A\,|$$

¶

Example 44 *similarity transformation*

$$\begin{aligned} |\,B^{-1}AB - \lambda I\,| &= |\,B^{-1}AB - B^{-1}\lambda IB\,| \\ &= |\,B^{-1}(A - \lambda I)B\,| \\ &= |\,B^{-1}\,|\,|\,A - \lambda I\,|\,|\,B\,| \\ &= |\,A - \lambda I\,| \end{aligned}$$

Definition 53 *matrix trace*
Let A be a square, $n \times n$ matrix. Then the trace of A is,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

Definition 54 *matrix transpose*
The *transpose* of a matrix

$$A = \{a_{ij}\}$$

is

$$A^T = \{a_{ji}\}$$

Definition 55 *complex conjugate*
The *complex conjugate* of a matrix

$$A = \{a_{ij}\}$$

is

$$\bar{A} = \{\bar{a}_{ij}\}$$

where $\bar{a} = p - qi$ if $a = p + qi$.

Definition 56 *big-O notaton*
Let $\phi(n)$ or $\phi(x)$ be a positive function, and let $f(n)$ or $f(x)$ be any function. Then $f = O(\phi)$ if $|f| < A\phi$ for some constant A and all values of n and x. Here O is called the *big-O* notation which denotes asymptoticity. The notation $f = O(\phi)$ is read, 'f is of order ϕ'.

Theorem 25 *properties of determinant*
Some other properties of the determinant are,

$$|\,A\,| = |\,A^T\,|$$

$$|\,\bar{A}\,| = \overline{|\,A\,|}$$

$$|\,I + \epsilon A\,| = 1 + \mathrm{Tr}(A) + O(\epsilon^2), \text{ for } \epsilon \text{ small}$$

Example 45 *notes on determinants*
For a square matrix A,

a. switching rows changes the sign of the determinant
b. factoring out scalars from rows and columns leaves the value of the determinant unchanged
c. adding rows and columns together leaves the determinant's value unchanged
d. multiplying a row by a constant c gives the same determinant multiplied by c
e. if a row or a column is zero, then the determinant is zero
f. if any two rows or columns are equal, then the determinant is zero

Theorem 26 *matrix trace*
Some properties of matrix trace are,

$$\mathrm{Tr}(A) = \mathrm{Tr}\left(A^T\right)$$

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$

$$\mathrm{Tr}(\alpha A) = \alpha\,\mathrm{Tr}(A)$$

Problem 23 *matrix transpose*
Prove that,

$$\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$$

Theorem 27 *property of matrix transpose*

$$(AB)^T = B^T A^T$$

Proof.

$$\begin{aligned}\left(B^T A^T\right)_{ij} &= \left(b^T\right)_{ik}\left(a^T\right)_{kj}\\ &= b_{ki}a_{jk}\\ &= a_{jk}b_{ki} = (AB)_{ji} = (AB)^T_{ij}\end{aligned}$$

¶

Definition 57 *matrix inverse*

Let A be a square matrix. Then the *inverse* of A, if it exists, is A^{-1} such that,

$$AA^{-1} = I$$

Furthermore, A is said to be *nonsingular* or *invertible* if its inverse exists, otherwise it is said to be *singular.*

Example 46 *matrix inverse*
For a 2 × 2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse of A is,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If A is a 3 × 3 matrix, then the inverse of A is,

$$A^{-1} = \frac{1}{|A|} \{\det(m_{ij})\}$$

where m_{ij} is a minor of A.
If A is an $n \times n$ matrix, then A^{-1} can be found by numerical methods, for example Gauss-Jordan elimination, Gaussian elimination, and LU decomposition.

Example 47 *finding matrix inverse*

The *Gaussian elimination* procedure solves the matrix equation $A\mathbf{x} = \mathbf{b}$ by first forming an augmented matrix equation $[A\,\mathbf{b}]$ and then transform this into an upper triangular matrix $\left[\{a'_{ij}\}\,\mathbf{b}'\right]$, where a'_{ij} are all zero except when $i \leq j$. Then,

$$x_i = \frac{1}{a'_{ii}}\left(b'_i - \sum_{j=i+1}^{k} a'_{ij}x_j\right)$$

The *Gauss-Jordan elimination* procedure finds matrix inverse by first forming a matrix $[A\,I]$, and then use the Gaussian elimination to transform this matrix into $[I\,B]$. The result matrix B is in fact A^{-1}.

The *LU decomposition* forms from the matrix A a product LU of two matrices, one lower- while the other upper triangular. This gives us three types of equation to solve, namely when $i < j$, $i = j$ and $i > j$, where i and j are the indices of row and respectively column of the matrix product. Then,

$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$$

Letting $\mathbf{y} = U\mathbf{x}$ we have $L\mathbf{y} = \mathbf{b}$, and therefore,

$$\begin{aligned} y_1 &= \frac{b_1}{l_{11}} \\ y_i &= \frac{y}{l_{ii}} \left(b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right) \end{aligned}$$

where $i = 2, \ldots, n$.

Then since $U\mathbf{x} = \mathbf{y}$,

$$x_n = \frac{y_n}{u_{nn}}$$

$$x_i = \frac{1}{n_{ii}} \left(y_i - \sum_{j=i+1}^{n} u_{ij} x_j \right)$$

where $i = n - 1, \ldots, 1$.

Theorem 28 *matrix inverse*
Let A and B be two square matrices of equal size. Then,

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof. Let $C = AB$. Then $B = A^{-1}C$ and $A = CB^{-1}$, therefore,

$$C = AB = \left(CB^{-1}\right)\left(A^{-1}C\right) = CB^{-1}A^{-1}C$$

Hence $CB^{-1}A^{-1} = I$, and thus $B^{-1}A^{-1} = (AB)^{-1}$. ¶

Definition 58 *Einstein's summation*

The *Einstein's summation* is the simplification of notation by omitting a summation sign, keeping in mind that repeated indices are implicitly summed over, for example $\sum_i a_{ik}a_{ij}$ becomes

$$a_{ik}a_{ij}$$

and $\sum_i a_i a_i$ becomes

$$a_i a_i$$

Definition 59 *matrix multiplication*

The multiplication of two matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ is the matrix $C = AB$ such that

$$c_{ik} = a_{ij}b_{jk}$$

Theorem 29 *associativity of matrix multiplication*
The matrix multiplication is associative.
Proof.

$$\begin{aligned}[(ab)c]_{ij} &= (ab)_{ik}c_{kj} = (a_{il}b_{lk})\,c_{kj} \\ &= a_{il}\,(b_{lk}c_{kj}) = a_{il}(bc)_{lj} = [a(bc)]_{ij}\end{aligned}$$

¶

Example 48 *matrix multiplication*
From Theorem 29, which shows us the associativity of matrix multiplication, we could write the multiplication of three matrices as $[abc]_{ij}$, which is the same as writing $a_{il}b_{lk}c_{kj}$. And this applies in a similar manner to the multiplication of four or more matrices.

Theorem 30 *non-commutativity of matrix multiplication*
If A and B are two square and diagonal matrices, then

$$AB = BA$$

But in general matrix multiplication is not commutative.

Definition 60 *block matrix*
A *block matrix* is a matrix which is is made up of small matrices put together, for example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A, B, C and D are matrices.

Theorem 31 *block matrix multiplication*

Block matrices may be multiplied together in the usual manner, for example,

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{bmatrix}$$

provided that all the products involved are possible.

Definition 61 *diagonal matrix*

Let $A = \{a_{ij}\}$ be an $n \times n$ matrix. Then A is called a *diagonal matrix* if $a_{ij} = 0$ when $i \neq j$. Here $1 \leq i, j \leq n$. In other words, a diagonal matrix has its components in the form $a_{ij} = c_i\delta_{ij}$, where c_i is a constant and δ_{ij} is the Kronecker delta,

$$\delta = \begin{cases} 0, \, i \neq j \\ 1, \, i = j \end{cases}$$

Theorem 32 *matrix diagonalisation*
A square matrix A can be diagonalised by the transformation

$$A = PDP^{-1}$$

where P is made up of the eigenvectors of A and D is the diagonal matrix desired.

Example 49 *matrix diagonalisation*
Matrix diagonalisation can greatly help reducing the number of parameters in a system of equations. For instance, the systems $A\mathbf{x} = \mathbf{y}$ when diagonalised becomes

$$PDP^{-1}\mathbf{x} = \mathbf{y}$$

that is $D\mathbf{x}' = \mathbf{y}'$, where $\mathbf{x}' = P^{-1}\mathbf{x}$ and $\mathbf{y}' = P^{-1}\mathbf{y}$. In this case, if A is an $n \times n$ matrix, we say that our new system obtained through the process of diagonalisation has canonicalised from $n \times n$ to n parameters.

Definition 62 *symmetric matrix*
A *symmetric* matrix is a square matrix A which satisfies

$$A^T = A$$

Example 50 *symmetric matrix*
If A is a symmetric matrix, then

$$A^{-1}A^T = I$$

Definition 63 *orthogonal matrix*
Let A be a square matrix. Then A is said to be *orthogonal* if

$$AA^T = I$$

Example 51 *orthogonal matrix*
Definition 63 is the same as saying that

$$A^{-1} = A^T$$

Theorem 33 *symmetric matrix*
A matrix A is symmetric if it can be expressed as

$$A = QDQ^T$$

where Q is an orthogonal matrix and D is a diagonal matrix.

Example 52 *symmetric matrix*
Any square matrix A may be decomposed into two terms added together, that is $A_s + A_a$ where A_s is a symmetric matrix and A_a an antisymmetric matrix, called respectively a *symmetric part* and an *antisymmetric part* of A. Furthermore,

$$A_s = \frac{1}{2}\left(A + A^T\right)$$

and,

$$A_a = \frac{1}{2}\left(A - A^T\right)$$

Definition 64 *inverse matrix*
Let A be a square, nonsingular matrix. Then the *inverse matrix* A^{-1} of A is a unique matrix for which,

$$AA^{-1} = I = A^{-1}A$$

Example 53 *finding the inverse of a matrix*
An inverse matrix may be found using the formula,

$$A^{-1} = \frac{1}{|A|}\,\mathrm{Adj}\,A$$

Example 54 *solving linear equations with the inverse*
Matrix equations of the form

$$A\mathbf{x} = \mathbf{b}$$

can be solved with the help of the inverse matrix A^{-1} as

$$\mathbf{x} = A^{-1}\mathbf{b}$$

where A is an $n \times n$ matrix, $\mathbf{x}$ a vector of size n whose components are variables, and $\mathbf{b}$ a vector of size n containing constants.

Theorem 34 *Cramer's rule*
Let A be the coefficient matrix and A_i a matrix formed from A by replacing the column of coefficients of x_i with the column vector of constants. Cramer's rule solves a system of linear equations through the use of determinants as follows.

$$x_i = \frac{|\,A_i\,|}{|\,A\,|}$$

Definition 65 *Jacobian determinant*
Let a system of n functions not necessarily linear be

$$\begin{aligned} y_1 &= f_1\left(x_1, \ldots, x_n\right) \\ &\vdots \\ y_n &= f_n\left(x_1, \ldots, x_n\right) \end{aligned}$$

Then a *Jacobian determinant* comprises all the first-order partial derivatives of the system arranged in ordered sequence, that is

$$\mid J \mid = \left| \frac{\partial y_1, \ldots, \partial y_n}{\partial x_1, \ldots, \partial x_n} \right| = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

Theorem 35 *functional dependence from Jacobian determinant*
Let a system of n equations be

$$y_i = f_i\left(x_1, \ldots, x_n\right)$$

$i = 1, \ldots, n$.

If $\mid J \mid = 0$, then y_i are functionally dependent.

On the other hand if $\mid J \mid \neq 0$, then y_i are functionally independent.

Definition 66 *Hessian*
A determinant $\mid H \mid$ composed of all the second-order partial derivatives, with the direct partials on the principal diagonal and the cross partials off the same, is called a *Hessian.* In other words, let a multivariable function be

$$z = f(x, y)$$

Then the Hessian of z is

$$|\,H\,| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix}$$

where $z_{xy} = z_{yx}$. Moreover, the *first principal minor* is

$$|\,H_1\,| = z_{xx}$$

and the *second principal minor* is

$$|\,H_2\,| = \begin{vmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{vmatrix} = z_{xx}z_{yy} - (z_{xy})^2$$

Theorem 36 *optimality of a multivariable function*
Let a multivariable function be

$$z = f(x, y)$$

and let the first-order conditions

$$z_x = z_y = 0$$

are met. Then a sufficient condition for z to be at optimum is

$$z_{xx}z_{yy} > (z_{xy})^2$$

together with

$$z_{xx}, z_{yy} < 0$$

in case of a maximum and

$$z_{xx}, z_{yy} > 0$$

in case of a minimum.

Definition 67 *definiteness*

From Definition 66,

if $|\, H_1 \,| > 0$ and $|\, H_2 \,| > 0$ the Hessian $|\, H \,|$ is said to be *positive definite*, and the second-order conditions for the minimum are met.

If $|\, H_1 \,| < 0$ and $|\, H_2 \,| > 0$ it is said to be *negative definite*, and the second-order conditions for the maximum are met.

Algorithm 1 *Procedure to test for the optimality of multivariable functions of two variables.*

```
$z = f(x, y)$
find $z_x$ and $z_y$
if $z_x = 0$ and $z_y = 0$ then
   find $z_{xx}$, $z_{xy}$ and $z_{yy}$
   find $H_1$ and $H_2$
   if $|\, H_1 \,| > 0$ and $|\, H_2 \,| > 0$ then
      $|\, H \,|$ is positive definite
   elseif $|\, H_1 \,| < 0$ and $|\, H_2 > 0 \,|$ then
      $|\, H \,|$ is negative definite
   endif
endif
```

Definition 68 *general Hessian*

Let $y = f(x_1, \ldots, x_n)$ be function of n variables. Then the n^{th}-order Hessian for this function is

$$|H| = \begin{vmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{vmatrix}$$

Then the *first principal minor* $|H_1|$ is simply x_{11}, and the i^{th} principal minor is

$$|H_i| = \begin{vmatrix} y_{11} & \cdots & y_{1i} \\ \vdots & \ddots & \vdots \\ y_{i1} & \cdots & y_{ii} \end{vmatrix}$$

Theorem 37 *positive definiteness through Hessian*

Let $y = f(x_1, \ldots, x_n)$ be function of n variables. Let the Hessian of y be represented by $|H|$.

Then if all the principal minors of $|H|$ are positive, then $|H|$ is positive definite and the second-order conditions for a relative minimum are met.

If the sign of the principal minors alternates between negagive and positive, then $|H|$ is negative definite and the second-order conditions for a relative maximum are met.

Example 55 *positive definiteness for three dimensions*

For

$$y = f(x_1, x_2, x_3)$$

the third-order Hessian is

$$|\,H\,| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

where

$$y_{11} = \frac{\partial^2 y}{\partial x_1^2},\ y_{12} = \frac{\partial^2 y}{\partial x_2 \partial x_1},\ \text{and so on}$$

The first-, second- and third-order Hessian's are respectively

$$|\,H_1\,| = y_{11},\ |\,H_2\,| = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

and

$$|\,H_3\,| = \begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix}$$

If $|\,H_1\,| > 0$, $|\,H_2\,| > 0$ and $|\,H_3\,| > 0$, then H is positive definite and the second-order condition for minimum is fulfilled.

If $|\,H_1\,| < 0$, $|\,H_2\,| > 0$ and $|\,H_3\,| < 0$, then $|\,H\,|$ is negative definite and the second-order condition for maximum is satisfied.

Definition 69 *discriminant*

A *discriminant* is a determinant of a quadratic form. Let the quadratic form be

$$z = ax^2 + bxy + cy^2$$

which is in matrix form

$$z = [\,x \quad y\,] \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Then the discriminant is

$$|\,D\,| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix}$$

The *first principal minor* of the discriminant is

$$|\,D_1\,| = a$$

and the *second principal minor*

$$|\,D_2\,| = \begin{vmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{vmatrix} = ac - \frac{b^2}{4}$$

Theorem 38 *definiteness of a function by the discriminant*
Let a quadratic form be

$$z = ax^2 + bxy + cy^2$$

and let the discriminant of z be $|\,D\,|$.

If $|\,D_1\,| > 0$ and $|\,D_2\,| > 0$, then $|\,D\,|$ is positive definite and $z > 0$ for all $x, y \neq 0$.

If $|\,D_1\,| < 0$ and $|\,D_2\,| > 0$, then $|\,D\,|$ is negative definite and $z < 0$ for all $x, y \neq 0$.

Theorem 39 *constrained optimisation with Lagrange multipliers*
Let

$$f(x, y)$$

be a function subject to a constraint

$$g(x, y) = k$$

where k is a constant. Then the optimisation of f can be done by first transforming f together with g into a new function

$$F(x, y, \lambda) = f(x, y) + \lambda\,(k - g(x, y))$$

and then solve the following equations,

$$\begin{aligned} F_x(x, y, \lambda) &= 0 \\ F_y(x, y, \lambda) &= 0 \\ F_\lambda(x, y, \lambda) &= 0 \end{aligned}$$

to obtain the critical values x_0, y_0 and λ_0 at which F and hence f are optimised.

Definition 70 *constrained optimisation*

In the constrained optimisation with Lagrange multipliers in Theorem 39 above, f is called an

objective or *origin function*

and F the

Lagrangian function

Definition 71 *bordered Hessian*

Let

$$f(x_1, \ldots, x_n)$$

be a function of n variables subject to constraints

$$g(x_1, \ldots, x_n)$$

Let

$$F(x_1, \ldots, x_n, \lambda) = f(x, \ldots, x_n) + \lambda \left(k - g(x_1, \ldots, x_n)\right)$$

Then the *bordered Hessian* $|\,\bar{H}\,|$ is defined as either

$$|\,\bar{H}\,| = \begin{vmatrix} F_{11} & F_{12} & \cdots & F_{1n} & g_1 \\ F_{21} & & & & g_2 \\ \vdots & & \ddots & & \vdots \\ F_{n1} & & & F_{nn} & g_n \\ g_1 & g_2 & \cdots & g_n & 0 \end{vmatrix}$$

or

$$|\,\bar{H}\,| = \begin{vmatrix} 0 & g_1 & \cdots & g_n \\ g_1 & F_{11} & & F_{1n} \\ \vdots & & \ddots & \vdots \\ g_n & F_{n1} & \cdots & F_{nn} \end{vmatrix}$$

This is simply the Hessian

$$\begin{vmatrix} F_{11} & \cdots & F_{1n} \\ \vdots & \ddots & \vdots \\ F_{n1} & \cdots & F_{nn} \end{vmatrix}$$

bordered by the first derivatives of the constraint with zero on the principal diagonal. The order of a bordered principal minor being determined by the order of the principal minor being bordered,

$$|\,\bar{H}\,| = |\,\bar{H}_n\,|$$

since in this case an $n \times n$ principal minor is being bordered.

Theorem 40 *bordered Hessian*
Let $f(x_1, \ldots, x_n)$ be a function of n variables subject to constraints

$$g(x_1, \ldots, x_n)$$

Let $|\bar{H}|$ be the bordered Hessian defined in Definition 71.

Then if $|\bar{H}_2|, \ldots, |\bar{H}_n| < 0$, then the bordered Hessian $|\bar{H}|$ is positive definite, and therefore is a sufficient condition for a minimum.

If $|\bar{H}_2| > 0$ $|\bar{H}_3| < 0$, $|\bar{H}_4| > 0$, and so alternatingly on, then $|\bar{H}|$ is negative definite, which is a sufficient condition for a maximum.

Example 56 *constrained optimisation using bordered Hessian*
Let $f(x, y)$ be a function to be optimised subject to a constraint $g(x, y) = k$, where k is a constant. Then the Lagrangian function becomes

$$F(x, y, \lambda) = f(x, y) + \lambda\,(k - g(x, y))$$

The first-order conditions for optimisation are

$$F_x = F_y = F_\lambda = 0$$

The second-order conditions for optimisation can be expressed together as a bordered Hessian

$$|\bar{H}| = \begin{vmatrix} F_{xx} & F_{xy} & g_x \\ F_{yx} & F_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix}$$

or

$$|\bar{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

Note 1 *first- and second-order conditions for optimisation of a function subject to some constraints*

Theorem 39 gives the first-order conditions for optimising a function subject to some constraints. Theorem 40 gives the second-order conditions for optimising a function subject to some constraints.

Definition 72 *Marshallian demand function*

A *Marshallian demand function* gives an expression of the amount of a good that a consumer will buy as a function of commodity prices and income available. It is derived by maximising the utility subjected to a budgetary constraint.

Example 57 *derivation of a Marshallian demand function*

Let a utility be

$$u = q_1 q_2$$

which is subject to a constraint

$$p_1 q_1 + p_2 q_2 = b$$

where b is the amount of income available, that is to say, our budget. Then the Lagrangian function is

$$U = q_1 q_2 + \lambda \left(b - p_1 q_1 - p_2 q_2 \right)$$

The first partial derivatives are then

$$u_1 = q_2 - \lambda p_1 = 0 \tag{18}$$

$$u_2 = q_1 - \lambda p_2 = 0 \tag{19}$$

$$u_\lambda = b - p_1 q_1 - p_2 q_2 = 0 \tag{20}$$

where u_1, u_2 are respectively u_{q_1} and u_{q_2}.

Simultaneously solving Equation's 18, 19 and 20 leads us to

$$\frac{q_2}{p_1} = \lambda = \frac{q_1}{p_2}$$

Hence $q_2 = q_1 p_1 / p_2$ and $q_1 = q_2 p_2 / p_1$ and from Equation 20 we have,

$$b = p_1 q_1 + p_2 \frac{p_1 q_1}{p_2} = p_2 q_2 + p_1 \frac{p_2 q_2}{p_1}$$

which yield us, for q_1 and q_2, the Marshallian demand functions which maximise satisfaction of the consumer subject to income and prices.

Next, we test the second-order conditions by firstly finding $u_{11} = 0$, $u_{22} = 0$, $u_{12} = u_{21} = 1$, $g_1 = p_1$ and $g_2 = p_2$, which give us

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & p_1 \\ 1 & 0 & p_2 \\ p_1 & p_2 & 0 \end{vmatrix}$$

which gives $|\bar{H}_2| = 2p_1p_2 > 0$ Hence $|\bar{H}|$ is negative definite and thus u is maximised.

Definition 73 *input-output analysis*

The production process of producing one good usually requires the input of many other *intermediate goods*. Let x_i be the total demand for product i, and let b be the final demand for the product from the ultimate users. Then,

$$x_i = a_{i1}x_1 + \ldots + a_{in}x_n + b_i$$

for $i = 1, \ldots, n$, where a_{ij} is a *technical coefficient* which represents the value of input i required to produce one monetary unit's worth of product j.

If we consider the total demand for every one of the products, then

$$\mathbf{x} = A\mathbf{x} + \mathbf{b}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

It follows from this that

$$\mathbf{x} = (I - A)^{-1}\mathbf{b}$$

The matrix A is known as the *matrix of technical coefficients.* It is also known as the *input-output table*, the rows being the inputs and the columns the outputs.. The matrix $I - A$ is known as the *Leontief matrix.*

Example 58 *complete input-output table*

In a complete input-output table, labour and capital would also be included as inputs. These give the value added by the firm. They are normally put as an extra row at the bottom of the matrix of technical coefficients A. The vertical summation of each column of the table is then equal to 1.

Definition 74 *eigenvalue and eigenvector*

Let A be a square matrix. Then a scalar λ such that the equation

$$A\mathbf{v} = \lambda\mathbf{v} \tag{21}$$

holds for some vector $\mathbf{v} \neq \mathbf{0}$ is called an *eigenvalue* † of A, and the vector $\mathbf{v}$ is called an *eigenvector* of A corresponding to the eigenvalue λ. The eigenvalue λ is also known as the *characteristic root*, or the *latent root*, while the eigenvector is also known as the *characteristic vector*, or the *latent vector*.

Note 2 *eigenvalue*

From Equation 21 it follows directly that

$$(A - \lambda I)\mathbf{v} = 0 \tag{22}$$

Then $A - \lambda I$ is called the *characteristic matrix* of A. Since $\mathbf{v}$ assumes a unique value and by assumption $\mathbf{v} \neq 0$, it follows that $A - \lambda I$ must be singular, which means that its rows must be a multiple of one another. Now $A - \lambda I$ is zero if and only if the *characteristic determinant* $|\, A - \lambda I \,|$ of A is zero.

In other words

$$|\, A - \lambda I \,| = 0 \tag{23}$$

which is called the *characteristic equation* of A. With Equation 23 there will be an infinite solution for $\mathbf{v}$ in Equation 22. In particular, if $\mathbf{v}$ is a solution, that is if it is an eigenvector, so is $k\mathbf{v}$ for any $k \neq 0$. We force a unique solution by using the *normalisation*

$$\sum v_i^2 = 1$$

Then the sign-definiteness of A can be determined from the characteristic roots λ's.

Thus if all λ's are positive, then A is positive definite; and if negative, negative definite.

Let at least one λ be zero, which is neither positive nor negative, if all the remaining λ's are nonnegative, then A is positive semidefinite; and if they are nonpositive, negative semidefinite.

Lastly, if some of the λ's are positive while others are negative, then A is indefinite.

Note 3

We have seen in Note 2 how, having found λ_i, where $i = 1, \ldots, n$, we find through normalisation the corresponding, unique $\mathbf{v}_i$. On the other hand if we have found first the $\mathbf{v}_i$'s, their corresponding λ_i's may be found by first forming a *transformation matrix*

$$T = [\mathbf{v}_1 \ldots \mathbf{v}_n]$$

and then the corresponding eigenvalues or the characteristic roots are obtained from

$$T^T A T = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & \lambda_n \end{bmatrix}$$

Definition 75

The vector equation, Equation 21, has as its solutions the zero vector $\mathbf{v} = 0$ together with all the corresponding eigenvalue-eigenvector pairs. The set of all the eigenvalues of A is called the *spectrum* of A. The *spectral radius* of A is then the largest of all the absolute values of the eigenvalues of A, that is to say,

$$\max_i |\lambda_i|$$

The set of all eigenvectors $\mathbf{v}_{ij}$, together with $\mathbf{0}$, forms a vector space called the *eigenspace* of A corresponding to λ_i.

Definition 76 *optimisation*

A problem of *optimisation* is one in which one tries to maximise or minimise a certain quantity called the *objective*, which depends on a finite number of variables. These may be either independent or related to one another through some *constraints.*

Definition 77 *mathematical programme*

A *mathematical programme* is an optimisation problem in which the objective and the constraints are given as functions or mathematical relationship. In other words,

$$\begin{aligned} \text{optimise:} \quad & z = f(x_1, \dots, x_n) \\ \text{subject to:} \quad & g_i(x_1, \dots, x_n) \left\{ \begin{matrix} \leq \\ = \\ \geq \end{matrix} \right\} b_i, \quad i = 1, \dots, m \end{aligned}$$

Some constraints are explicitly stated as requirements, others are hidden conditions. These latter need to be pin-pointed through the study and understanding of the model and its inputs.

Definition 78 *linear programme*

A *linear programme* is a mathematical programme all the functions involved of which are linear. This means that,

$$f(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$$
$$g_i(x_1, \ldots, x_n) = a_{i1}x_1 + \cdots + a_{in}x_n$$

where $i = 1, \ldots, m$ and c_j and a_{ij}, $j = 1, \ldots, n$, are constants. If there is an additional restriction on the input variables that they be all integers, then the optimisation problem is called an *integer programme.* A mathematical programme which is not a linear programme is said to be *nonlinear.*

Definition 79 *quadratic programme*

A *quadratic programme* is a mathematical programme in which all the constraints are linear and the objective function is in quadratic form, which is in general,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_ix_j + \sum_{i=1}^{n} d_ix_i$$

where c_{ij} and d_i are constants.

Definition 80 *standard form*

A linear programme is said to be in *standard form* if all the constraints are equalities and if one feasible solution is known. In other words, our problem is now

$$\begin{aligned} \text{optimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

Definition 81 *initial feasible solution*

One may change any linear programme into the standard form by adding a *slack variable* to the left-hand side of a constraint of the form $\sum a_{ij}x_j \leq b_i$ to obtain

$$\sum_{j=1}^{n} a_{ij}x_j + x_{p_k} = b_i$$

where $p_k > n$ and $k = 1, 2, \ldots$.

Similarly one may add a *surplus variable* to the right-hand side of a constraint of the form $\sum a_{ij}x_j \geq b_i$ to obtain $\sum a_{ij}x_j = b_i + x_{q_l}$

$$\sum_{j=1}^{n} a_{ij}x_j - x_{q_l} = b_i$$

where $q_l > n$ and $l = 1, 2, \ldots$. Next, all the slack and surplus variables are added to the objective function with zero coefficients.

Then if we add an *artificial variable* to the left-hand side of each constraint where there is no slack variable, then the *initial feasible solution* is $\mathbf{x}_0 = \mathbf{b}$, where $\mathbf{x}$ is the vector of slack and artificial variables. The artificial variables are added to the objective function with a large negative coefficient $-M$.

Definition 82 *linear dependence*

A set of n vectors of m dimensions $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is said to be *linearly dependent* if there exist some constants $\alpha_1, \ldots, \alpha_n$ not all of which are zero, such that

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = \mathbf{0} \tag{24}$$

It is said to be *linearly independent* if the condition in Equation 24 implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Theorem 41

Consider a set of n vectors of m dimensions. If $n > m$, then the set is linearly dependent.

Definition 83 *convex combination*

A vector $\mathbf{v}$ is called a *convex combination* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ if there exist some nonnegative constants $\beta_1, \ldots, \beta_n$, where

$$\beta_1 + \cdots + \beta_n = 1$$

such that

$$\mathbf{v} = \beta_1 \mathbf{v}_1 + \cdots + \beta_n \mathbf{v}_n$$

Definition 84 *convex set*

A set of m-dimensional vectors is said to be *convex* if for any two vectors belonging to the set the line segment between them also belongs to the set.

Theorem 42

All points on the line segment joining any two vectors may be expressed as a convex combination of the two vectors.

Definition 85 *extreme point*

A vector **v** is called an *extreme point* of a convex set if it can not be expressed as a convex combination of two other vectors in the set.

In other words, an extreme point of a convex set K is a point x in K that cannot be written as $x = \theta y + (1 - \theta)z$ with $0 < \theta < 1$, y and z in K, and $y \neq z$, that is to say, an extreme point is a point which is not an *interior point* of any line segment belonging to K.

An equivalent definition of an extreme point is that x is an extreme point of a convex set K if $K\setminus\{x\}$ is convex.

Definition 86 *bounded set*

A *metric space* is a non-empty set X for which is defined a concept of distance. The *distance* d is called a *metric* on X, having such properties that, for any points x and y in X, we have $d(x, y) \geq 0$, and $d(x, y) = 0$ implies

$$x = y$$

Furthermore,

$$d(x, y) = d(y, x)$$

and

$$d(x, y) \leq d(x, z) + d(z, y)$$

Let X be a metric space with metric d, let A be a subset of X and let x be any point of X. Then the *distance from x to A* is defined as

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

whereas the *diameter of A* is defined as

$$d(A) = \sup \{d(a_1, a_2) : a_1 \text{ and } a_2 \in A\}$$

Then a set is said to be *bounded* if its diameter is finite.
Further, let x_0 be a point in X and r a positive real number. Then the *open sphere* $S_r(x_0)$ with *centre* x_0 and *radius* r is the subset of X defined by

$$S_r(x_0) = \{x : d(x, x_0) < r\}$$

A point x in X is called a *limit point* of A if each open sphere centred on x contains at least one point of A different from x. A subset F of X is said to be *closed* if it contains all its limit points.

Definition 87 *linear space*

A *linear space*, aka a *vector space*, is a non-empty set L on which is defined two binary processes, say *addition* and *scalar multiplication*. Addition is defined such that for any x, y and z in L, then $x + y$ is again in L;

$$x + y = y + x$$

$$x + (y + z) = (x + y) + z$$

there exists a unique *identity* element 0, aka *zero element* or the *origin*, such that $x + 0 = x$ for every x; and there exists a unique *inverse* element $-x$ for every x, such that $x + (-x) = 0$.

Scalar multiplication is defined with regard to *scalars*, some instances of which are real and complex numbers, such that for any scalar α and any x and y in L, αx is again in L

$$\alpha(x+y) = \alpha x + \alpha y$$

$$(\alpha+\beta)x = \alpha x + \beta x$$

$$(\alpha\beta)x = \alpha(\beta x)$$

and $1x = x$, where 1 is the identity for scalar multiplication.

A *normed linear space* is a linear space on which is defined a *norm*, that is a function which maps each element x in the space to a real number $||x||$ in such a manner that $||x|| \geq 0$, and $||x|| = 0$ if and only if $x = 0$

$$||x+y|| \leq ||x|| + ||y||$$

and

$$||\alpha x|| = |\alpha| \, ||x||$$

Theorem 43

A normed linear space is a metric space.

Theorem 44

Any vector in a closed and bounded convex set with a finite number of extreme points can be expressed as a convex combinations of the extreme points.

Definition 88 *definition of the problem*

For two vectors, that is points, x and y in $\mathbf{R}^n$, we write $\mathbf{x} \geq \mathbf{y}$ if and only if $x_i \geq y_i$ for all $1 \leq i \leq n$. A system of m weak linear inequalilties in n variables can be written as $A\mathbf{x} \geq \mathbf{b}$, where A is an $m \times n$ matrix.

A fundamental question concerning such system is whether it is *consistent*, that is to say, whether there exists some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$.

A system may be *inconsistent*, or it may have a set of solutions which is *unbounded.* If we sketch our problem on a graph, we may see that it's solution set is *convex.*

Theorem 45 *solution space*

The solution space of a set of simultaneous linear equations is a convex set the number of extreme points of which is finite.

Theorem 46 *extreme-point solution*

Let S be the set of all feasible solutions to the linear programme in standard form in Definition 80, in other words, S is the set of all vectors $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$, where A is an $m \times n$ matrix. Then S is a convex set, and the number of its extreme points is finite. The objective function attains its optimum, provided that one exists, at an extreme point of S. If $m \leq n$, then the extreme points of S have at least $n - m$ zero components.

Algorithm 2

Procedure for finding basic feasible solutions.:

```
Input: $A\mathbf{x} = \mathbf{b}$, $A$ is an $m \times n$ matrix, $m \le n$, $\operatorname{rank} A = m$
$[\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,] \leftarrow A$
$(x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}) \leftarrow (A\mathbf{x} = \mathbf{b})$
for $i = m + 1$ to $n$ do
  $x_i \leftarrow 0$
endfor
$(x_1, \ldots, x_n) \leftarrow$ solve $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$
```

Definition 89 *simplex method*

The *simplex method* is a matrix procedure which solves linear programmes of the standard form as described in Definition 80 where $\mathbf{b} > \mathbf{0}$. Starting from a basic feasible solution $\mathbf{x}_0$ we locate successively other basic feasible solutions giving better values for our objective. For minimisation programmes the method uses Table 3, for maximisation programmes the same table is also used but with the sign of entries in the bottom row reversed.

Table 3

Table used for minimisation programming in simplex method.:

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{c}^T - \mathbf{c}_0^T A$	$-\mathbf{c}_0^T \mathbf{b}$

Table 4 *Description of the simplex method.*

```
while negative number exists in d do
    Locate the most negative number in the bottom row of the simplex
        table, excluding the last column. The column in which we find
        this number is called the work column. If more than one such
        column exist, choose one of them.
```

Find the smallest of the ratios between the elements in the last column and the elements in the work column of the same row, if these latter are positive. The element in the work column that yields this smallest ratio is called the *pivot element*. If there are more than one of these, choose one. If none of the elements in the work column is positive, the programme has no solution.

Using elementary row operations, convert the pivot element to 1 and reduce all other elements in the work column to 0.

Replace the x-variable in the pivot row and first column by the x-variable in the first row and pivot column. This new first column then becomes the current set of basic variables.

endwhile

The optimal solution is one in which all the basic variables assume the corresponding values in the last column, while the remaining variables are zero. The optimal value of the objective function is then the value of the last row and last column for a maximisation programme, and the negative of this value if the programme is one of minimisation.

Algorithm 3 *Algorithm for the simplex procedure.*

```
$j \leftarrow 0$
while there exists a negative number in d do
    $j \leftarrow j + 1$
    for $i = 1$ to $n$ do
        {c's} ← (column no. of the most negative no. in the bottom row)
        (work column) ← choose one of the {c}'s
        $k \leftarrow$ (work column)
    endfor
    $\rho_{pivot} \leftarrow M$
    $c \leftarrow 0$
    $soln \leftarrow 0$
    **for** $i = 1$ to $m$ **do**
        **if** $(\mathbf{a}_j)_{ik} > 0$ **then**
            $soln \leftarrow 1$
            $\rho \leftarrow \frac{(\mathbf{b}_j)_i}{(\mathbf{a}_j)_{ik}}$
            **if** $\rho < \rho_{pivot}$ **then**
                $r \leftarrow i$
            **endif**
        **endif**
    **endfor**
    **if** $soln = 0$ **then**
        no solutions exist
    **endif**
```

```
    **convert**† $A$, such that $(\mathbf{a}_j)_{rk} = 1$ and $(\mathbf{a}_j)_{ik} = 0$, $1 \le i \le m$, $i \ne r$
    $(\mathbf{x}_0)_r \leftarrow x_k$
**endwhile**
**for** $i = 1$ to $m$ **do**
    $(\mathbf{x}_0)_i^* \leftarrow (\mathbf{b}_j)_i$
**endfor**
**for** $i = m + 1$ to $n$ **do**
    $x_i^* \leftarrow 0$
**endfor**
$z^* \leftarrow e_j$
**if** the programme is one of minimisation **then**
    $z^* \leftarrow -z^*$
**endif**
```

Definition 90 *two-phase method*

The *two-phase method* is a procedure modified from the simplex method to cope with cases when artificial variables exist in the initial solution $\mathbf{x}_0$, in order to minimise the round-off errors that occur in the calculation. The last row in Table 3 in this case is

$$\mathbf{d} = \mathbf{c}^T - \mathbf{c}_0^T A = \mathbf{d}_1 + M\mathbf{d}_2$$

and consequently we have Table 5 which is used here. Algorithm 3 is then firstly applied to the last row, and then again to those elements directly above the zeros in that row.

When an artificial variable is removed from the first column of the table, it ceases to be basic and may be removed from the top row of the table together with the entire column under it. When the last row contains only zeros, it may be deleted from the table. The programme has no solution if non-zero artificial variables are present in the final basic set.

Table 3 *Table used for minimisation programming using the two-phase method.*

		$\mathbf{x}^T$ $\mathbf{c}^T$	
$\mathbf{x}_0$	$\mathbf{c}_0$	A	$\mathbf{b}$
		$\mathbf{d}_1$ $\mathbf{d}_2$	$-\mathbf{c}_0^T\mathbf{b}$

Definition 91 *duality in linear programming*

Given a linear programme in the variables $x_1, \ldots, x_n$, there exists another linear programme associated with it, called its *dual*, which is in the variables $w_1, \ldots, w_m$. The original programme is called the *primal*. The primal completely determines the form of its dual.

The *symmetric dual* of a primal linear programme in the matrix form

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} \geq \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

is the linear programme

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} \quad & A^T\mathbf{w} \leq \mathbf{c} \\ \text{with:} \quad & \mathbf{w} \geq \mathbf{0} \end{aligned}$$

The dual variables $w_1, \ldots, w_m$ are known as *shadow costs.*

The *unsymmetric dual* of the primal

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} \quad & A^T\mathbf{w} \leq \mathbf{c} \end{aligned}$$

The unsymmetric dual of the primal

$$\begin{aligned} \text{maximise:} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to:} \quad & A\mathbf{x} = \mathbf{b} \\ \text{with:} \quad & \mathbf{x} \geq 0 \end{aligned}$$

is

$$\begin{aligned} \text{minimise:} \quad & z = \mathbf{b}^T\mathbf{w} \\ \text{subject to:} \quad & A^T\mathbf{w} \geq \mathbf{c} \end{aligned}$$

Note 4

We may see from Definition 91 that the dual of a programme in standard form is not itself in standard form. These duals are said to be *unsymmetric*.

Theorem 47

If an optimal solution exists for either the primal or the dual programme, then the other programme also has an optimal solution. If the duality is symmetric, then the two functions have the same optimal value. If the duality if unsymmetric, then the optimal value of each function can be derived from that of the other.

Definition 92 *cut algorithms*

Algorithms which change the boundary of the solution region in order to find the optimal solution of an integer programme are called *cut algorithms.*

The branch-and-bound algorithm does this by splitting the solution region into two and then discard the one which does not contain the optimal solution.

The Gomory algorithm, on the other hand, reduce the feasible region with the help of a new constraint without the region being splitted.

Definition 93 *branching*

We call *branching* a process by which a programme whose solution contains a non-integral

$$j < x_i < k$$

is made into two separate programmes having the additional constraint

$$x_i \leq j$$

in one, and

$$x_i \geq k$$

in the other, the objective together with all the constraints of the original problem of which remain the same. Here j and k are positive integers and $j < k$.

Definition 94 *bounding*

In the branch-and-bound algorithm, if the objective is maximisation, the value of the objective obtained when the first integral approximation occurs is said to be the lower bound for the problem, and if the objective is minimisation it is said to be the upper bound of the same.

Algorithm 4 *Branch-and-bound algorithm for integer programming.*

```
find first approximation
while approximations not all integers do
   choose $x_i$ from all non-integral variables such that
         $\min\left(|x_i - \lfloor x_i \rfloor|, |x_i - \lceil x_i \rceil|\right)$ is maximised
   branch
   choose the branch whose value of the objective
          is maximum
endwhile
solution←last approximation
```

Example 59 *branch-and-bound example*
(*Problem 6.9; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} \quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

Solve by branch-and-bound algorithm.

Draw a simplex table of Programme 1.

		x_1	x_2	x_3	x_4	
		1	2	1	0	
x_4	0	2	3	3	1	11
		-1	-2	-1	0	0

Replace x_4 for x_2 as the basic variable.

	x_1	x_2	x_3	x_4	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	$\frac{11}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	$\frac{22}{3}$

$x_2^* = \frac{11}{3} = 3.6$, $x_1^* = x_3^* = x_4^* = 0$, $z^* = \frac{22}{3}$

Since $3 < x_2^* < 4$, branch into two programmes, namely Programme 1 where $x_2 \leq 3$, and Programme 2 where $x_2 \geq 4$. Consider first Programme 2.

$$\begin{array}{rl} \text{maximise:} & z = x_1 + 2x_2 + x_3 \\ \text{subject to:} & 2x_1 + 3x_2 + 3x_3 \leq 11 \\ & x_2 \leq 3 \\ \text{with:} & \text{all variables non-negative and integral} \end{array}$$

Use the simplex method in a tabulated form.

		x_1	x_2	x_3	x_4	x_5	
		1	2	1	0	0	
x_4	0	2	3	3	1	0	11
x_5	0	0	1	0	0	1	3
		-1	-2	-1	0	0	0

Replace the basic variable x_5 with x_2.

	x_1	x_2	x_3	x_4	x_5	
x_4	2	0	3	1	-3	2
x_2	0	1	0	0	1	3
	-1	0	-1	0	2	6

Replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	1
x_2	0	1	0	0	1	3
	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	7

$x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, $z^* = 7$

Then consider Programme 3.

$$\begin{aligned}
\text{maximise:}\quad & z = x_1 + 2x_2 + x_3 \\
\text{subject to:}\quad & 2x_1 + 3x_2 + 3x_3 \leq 11 \\
& x_2 \geq 4 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}$$

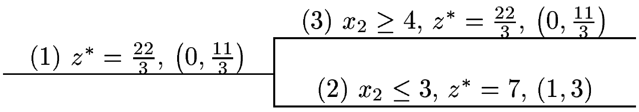
Draw a table for the two-phase method.

		x_1	x_2	x_3	x_4	x_5	x_6	
		1	2	1	0	0	$-M$	
x_4	0	2	3	3	1	0	0	11
x_6	$-M$	0	1	0	0	-1	1	4
		-1	-2	-1	0	0	0	0
		0	-1	0	0	1	-1	-15

Change x_4 for x_2 in the basic variables.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	$\frac{2}{3}$	1	1	$\frac{1}{3}$	0	0	$\frac{11}{3}$
x_6	$-\frac{2}{3}$	0	-1	$-\frac{1}{3}$	-1	1	$\frac{1}{3}$
	$\frac{1}{3}$	0	1	$\frac{2}{3}$	0	0	$\frac{22}{3}$
	$\frac{2}{3}$	0	1	$\frac{1}{3}$	1	-1	$-\frac{34}{3}$

The coefficient parts of the row corresponding to the basic variable x_6 and the last row cancel each other. The optimal result is $x_2^* = \frac{11}{3}$, $x_1^* = x_3^* = x_4^* = x_5^* = x_6^* = 0$ and $z^* = \frac{22}{3}$.



(3) $x_2 \geq 4$, $z^* = \frac{22}{3}$, $\left(0, \frac{11}{3}\right)$

(1) $z^* = \frac{22}{3}$, $\left(0, \frac{11}{3}\right)$

(2) $x_2 \leq 3$, $z^* = 7$, $(1, 3)$

Therefore the solution is $x_1^* = 1$, $x_2^* = 3$, $x_3^* = x_4^* = x_5^* = 0$, and $z^* = 7$.

#

Algorithm 5 *Gomory algorithm for integer programming.*

```
while solution not wholly all integers do
    choose one non-integral optimal approximation
    write a relation from the row where that variable is basic
    rewrite the relation to make all fractional coefficients
            some integer plus a proper fraction
    move all the fractions to LHS, and all the non-fractions
            to RHS
    write a new constraint as LHS $\geq 0$
    find the solution for the original problem together
            with the new constraint
endwhile
```

Example 60

(*Problem 7.1; Bronson, 1982*)

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 \\ \text{subject to:} \quad & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ \text{with:} \quad & x_1, x_2 \text{ non-negative and integral} \end{aligned}$$

Use cut algorithm.

Solve

Find the first approximation of Programme 1 normally using the simplex method.

		x_1	x_2	x_3	x_4	
		2	1	0	0	
x_3	0	2	5	1	0	17
x_4	0	3	2	0	1	10
		-2	-1	0	0	0

Since $\frac{10}{3} < \frac{17}{2}$, we know that 3 is the pivot element, and therefore we replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	$\frac{31}{3}$
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$
	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{20}{3}$

We have $x_1^* = \frac{10}{3}$, $x_3^* = \frac{31}{3}$, $x_2^* = x_4^* = 0$ and $z^* = \frac{20}{3}$.

Since both x_1^* and x_3^* are non-integers, arbitrarily choose the former to generate a new constraint. Then our Programme 2 becomes,

$$\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 &= \frac{10}{3} = 3 + \frac{1}{3} \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &= 3 - x_1 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &\geq 0 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 &\geq \frac{1}{3} \\
2x_2 + x_4 &\geq 1
\end{aligned}$$

and our new programme becomes

$$\begin{aligned}
\text{maximise:}\quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 \\
\text{subject to:}\quad & \frac{11}{3}x_2 - \frac{2}{3}x_4 = \frac{31}{3} \\
& x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_4 = \frac{10}{3} \\
& 2x_2 + x_4 \geq 1 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	
		2	1	0	0	0	$-M$	
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{10}{3}$
x_3	0	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{31}{3}$
x_6	$-M$	0	2	0	d1	-1	1	1
		-2	-1	0	0	0	0	0
		0	-2	0	-1	1	-1	-1

Now x_2 replaces x_6 in the basic variables and becomes the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	3
x_3	0	0	1	$-\frac{15}{6}$	$\frac{11}{6}$	$-\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	-2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	0	0	0	0

This becomes,

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	$\frac{1}{3}$	3
x_3	0	0	1	$-\frac{5}{2}$	$\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{13}{2}$

Then our first approximation of Programme 2 is $x_1^* = 3$, $x_2^* = \frac{1}{2}$, $x_3^* = \frac{17}{2}$, $x_4^* = x_5^* = 0$, and $z^* = \frac{13}{2}$.

Arbitrarily choose x_2^* to generate the new constraint.

$$\begin{aligned} x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 &= \frac{1}{2} \\ \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &= -x_2 \\ \frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &\geq 1 \\ x_4 - x_5 &\geq 1 \end{aligned}$$

Then our Programme 3 becomes,

$$\begin{aligned} \text{maximise:} \quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to:} \quad & x_1 + \frac{1}{3}x_5 = 3 \\ & x_3 - \frac{5}{2}x_4 + \frac{11}{6}x_5 = \frac{17}{2} \\ & x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = \frac{1}{2} \\ & x_4 - x_5 \geq 1 \\ \text{with:} \quad & \text{all variables non-negative and integral} \end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		2	1	0	0	0	0	$-M$	
x_1	0	1	0	0	0	$\frac{1}{3}$	0	0	3
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_3	0	1	0	0	$-\frac{5}{2}$	$\frac{11}{6}$	0	0	$\frac{17}{2}$
x_7	$-M$	0	0	0	1	-1	-1	1	1
		-2	-1	0	0	0	0	0	0
		0	0	0	-1	1	1	-1	-1

Then x_4 replaces the basic x_7 to become the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	1	0	0	0	$-\frac{2}{3}$	$-\frac{5}{2}$	11
x_4	0	0	0	1	-1	-1	1
	-2	-1	0	0	0	0	0

Next, x_1 remains basic and becomes a pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	-1	0	0	$\frac{1}{3}$	0	6

This becomes

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	6

The optimum point for Programme 3 is then, $x_1^* = 3$, $x_3^* = 8$, $x_4^* = 1$, $x_2^* = x_5^* = x_6^* = 0$ and $z^* = 6$.

Therefore the solution to the original problem Programme 1 is $x_1^* = 3$, $x_2^* = 0$ at the objective value $z^* = 6$.

#

Definition 95 *transportation problem*

A *transportation problem* involves m *sources* each of which supplies a_i, $i = 1, \ldots, m$, units of a certain product, and n *destinations* each of which requires b_i, $i = 1, \ldots, n$, units of the same. The problem may be stated as following.

$$\begin{aligned}
\text{maximise:}\quad & z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \\
\text{subject to:}\quad & \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1,\ldots,m \\
& \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1,\ldots,n \\
\text{with:}\quad & \text{all } x_{ij} \text{ non-negative and integral}
\end{aligned}$$

The total supply and the total demand are assumed to be equal. Were this not so, a fictitious destination or a fictitious source is added.

Definition 96 *north-west corner rule*

The *north-west corner rule* finds an initial basic solution for the transportation algorithm of the integer programming. It begins with the (1,1) cell in the $m \times n$ table, and allocates as many units as possible to x_{11} violating neither the constraints of supply, that is the summation along each row, nor those of demand, that is the summation along each column. Then carry on moving for each step either right or downwards, until we reach the lower-right corner, x_{mn}.

Definition 97 *loop*

A *loop*, which is a sequence of cells in the table used for finding the solution in the transportation problem, has the following properties.

a. each pair of consecutive cells is on either the same row or the same column

b. no three, or in fact any odd-numbered, consecutive cells lie in the same row or column

c. the first and the last cells are on the same row or column

d. the path along the loop is self-avoiding, that is no cells appear more than once in the sequence

Algorithm 6 *Transportation algorithm.*

```
while optimal solution not attained do
  find an initial, basic feasible solution using, for instance,
      the North-west corner rule
  let either $u_i = 0$ or $v_j = 0$ depending on whether
      the $i^{\text{th}}$-row or the $j^{\text{th}}$-column
      has the maximum number of basic solutions
  find all $u_i$ and $v_j$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$
      from $u_i + v_j = c_{ij}$ for basic variables, and
      from $c_{ij} - u_i - v_j$ for non-basic variables
  improve the solution
endwhile
```

Note 5 *optimum for transportation problem*
In a transportation problem, optimal solution is achieved when

$$c_{ij} - u_i - u_j \geq 0$$

for all transportation costs per unit c_{ij} of all non-basic variables.

Definition 98 *Present and future values*

The *present value* p_0 or the *principal* is the amount initially borrowed or invested. The *future value* p_t is the principal after a period of time t.

Definition 99 *Annual percentage rate*

Interest rates expressed per annum are called *nominal rates*, i. The *annual percentage rate* or *effective annual rate* i_a is the equivalent annual rate of different interest rates variously compounded.

Definition 100 *Sequence*

A *sequence* is a list of numbers which follows a definite pattern. It is called an *arithmetic sequence* if each of its terms is obtained from the term immediately preceding it by an addition of a constant d, which is called the *common difference*. It is called a *geometric sequence* if each of its term is obtained from the previous term by a multiplication of a constant r, the *common ratio*.

Definition 101 *Series*

A *series* is the sum of the terms of sequence. It is called a *finite series* is one whose number of terms is finite, otherwise it is called an *infinite series*. An *arithmetic series* or *arithmetic progression* is the sum of the terms of an arithmetic sequence. Likewise a *geometric series* or *geometric progression* is the sum of the terms of geometric sequence.

Theorem 48 *Arithmetic series*
The value of the n^{th} term of an arithmetic series is

$$T_n = a + (n-1)d$$

The sum of its first n terms is

$$S_n = \frac{n}{2}\left(2a + (n-1)\,d\right)$$

Theorem 49 *Geometric series*
The n^{th} term of a geometric series is

$$T_n = ar^{n-1}$$

The sum of the first n terms of it is

$$\begin{aligned} S_n &= a + ar + \cdots + ar^{n-1} \\ &= \frac{a\left(1-r^n\right)}{1-r} \\ &= \frac{a\left(r^n-1\right)}{r-1} \end{aligned}$$

When the number of terms approaches infinity, the summation in cases where $r < 1$ becomes

$$S_\infty = \frac{a}{1-r}$$

Definition 102 *Simple and compound interests*
A *simple interest* is a fixed percentage of the principal paid to an investor each year. A *compound interest* is an interest paid on the principal plus any interest accumulated in previous years.

Theorem 50 *Present value for simple interest*
The present value for simple interest is

$$p_t = p_0(1 + it)$$

where i is the interest rate and t the time in years.

Theorem 51 *Present value, compound interest*
The present value in the case of compound interest is

$$p_t = p_0(1+i)^t$$

Note 6 *Compounding more than once a year*
The interest may be compounded more than once a year, for example biannually, quarterly, monthly, weekly, daily, or continuously. Each time period is called a *conversion period* or *interest period*. The interest rate applied at each conversion is i/m, where m is the number of conversion periods per year. The number of conversion periods over t years is then $n = mt$.

Theorem 52 *Present value, compounding several times per year*
The present value at the end of n conversion periods is

$$p_t = p_0 \left(1 + \frac{i}{m}\right)^n = p_0 \left(1 + \frac{i}{m}\right)^{mt}$$

where all the variables and parameters are as previously defined.

Theorem 53 *Present value, continuous compounding*
When the number of compoundings per year becomes very large, the present value becomes

$$p_t = p_0 e^{it}$$

Proof. Since $p_t = p_0 \left(1 + \frac{i}{m}\right)^{mt}$ and $\lim_{m\to\infty} \left(1 + \frac{i}{m}\right)^m = e^i$, we have the proof. ¶

Theorem 54 *Annual percentage rate*
The annual percentage rate when compounding occurs m times per year is

$$i_a = \left(1 + \frac{i}{m}\right)^m - 1$$

Appendix

Course Outline

Week	*Date*	*Topic of lecture*	*Hours*
1	25 Oct 2005	Graph and derivative	3
2	1 Nov 2005	Calculus of multivariable functions	3
3	8 Nov 2005	Exponential, log and nonlinear functions	3
4	15 Nov 2005	Matrix	3
5	22 Nov 2005	Linear algebra	3
6	29 Nov 2005	Linear programming	3
7	6 Dec 2005	Integer programming	3
8	20 Dec 2005	Financial mathematics	3
9	10 Jan 2006	Integral calculus	3
10	17 Jan 2006	Integral calculus	3
11	24 Jan 2006	Simultaneous equations	3
12	31 Jan 2006	Differential equation	3
13	7 Feb 2006	Trigonometric functions and power series	3
14	14 Feb 2006	Differential equation	3

Materials

Content	*Date begun*	*Dated*	*Last updated*
Graph and derivative	20 Oct 05	25 Oct 05	10 Dec 05
Calculus of multivariable functions	20 Oct 05	1 Nov 05	18 Nov 05
Exponential-, logarithmic and nonlinear functions	28 Oct 05	8 Nov 05	18 Nov 05
Matrix	5 Nov 05	15 Nov 05	10 Dec 05
Linear algebra	5 Nov 05	22 Nov 05	10 Dec 05
Examples for linear algebra	17 Jan 06	31 Jan 06	31 Jan 06
Exercises for linear algebra	7 Feb 06	7 Feb 06	7 Feb 06
Linear programming	5 Nov 05	29 Nov 05	10 Dec 05
Examples for linear programming	5 Nov 05	6 Dec 06	10 Dec 06
Integer programming	5 Nov 05	6 Dec 05	11 Dec 05
Financial mathematics	5 Nov 05	13 Dec 05	20 Dec 05
Examples for financial mathematics	18 Feb 06	20 Feb 06	20 Feb 06
Simultaneous equations	5 Nov 05	24 Jan 06	26 Jan 06
Differential equation	5 Nov 05	31 Jan 06	31 Jan 06
Integral calculus	5 Nov 05	10 Jan 06	13 Feb 06
Integral calculus	6 Feb 06	7 Feb 06	7 Feb 06
Examples for integral calculus	17 Jan 06	7 Feb 06	7 Feb 06
Exercises for integral calculus	10 Jan 06	7 Feb 06	7 Feb 06
Difference equation	5 Nov 05	20 Feb 06	19 Feb 06
Examples for difference equation	19 Feb 06	20 Feb 06	20 Feb 06
Appendix			
Midterm examination	19 Dec 05	20 Dec 06	20 Dec 06
[solution]			5 Jan 06
Quiz 1	30 Jan 06	31 Jan 06	13 Feb 06
[solution]			16 Feb 06
Quiz 2	7 Feb 06	7 Feb 06	7 Feb 06
[solution]			23 Feb 06
Quiz 3	13 Feb 06	14 Feb 06	13 Feb 06
[solution]			14 Feb 06
Final examination	19 Feb 06	20 Feb 06	20 Feb 06
[solution]			22 Feb 06

Teaching method

There were lectures and practices in class. All practice exercises, quizzes and exams were done in an opened-book manner. Practice of problem and exercise in class were preferred to assignment and homework, since in the latter many students did not do the work themselves. In some of these practice the questions for the students were all different in order that they may start to think for themselves.

Teaching media

A camera projector and a microphone were the hardware media used. The material used as a media were lecture projections in this collection and lecture hand-outs, which will go into another collection. Because of the distance between the class and its lecturer, feedbacks of homework were sent to students by email. Textbooks on Calculus from the library were used in some of the exercise sessions, for the students to learn how to use books as a resource to help them solve problems and answer questions.

Evaluation methods

means	*per cent*
Attendance	10
Homework	10
Quiz 1	10
Quiz 2	10
Quiz 3	10
Midterm exam	20
Final exam	30

Evaluation will be based on the distribution plot, as relative performance among students.

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Quiz 1
Business Mathematics

31^{st} January 2006

Time: 1 hour (10–11pm)

1. Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

Find all the eigenvalues and a basis for each eigenspace. If possible, find invertible matrices P such that $P^{-1}AP$ is diagonal.

Solution. Form the characteristic matrix,

$$tI - A = \begin{bmatrix} t-3 & -1 & -1 \\ -2 & t-4 & -2 \\ -1 & -1 & t-2 \end{bmatrix}$$

Then,

$$\begin{aligned} |\,tI - A\,| &= \begin{vmatrix} t-3 & -1 & -1 \\ -2 & t-4 & -2 \\ -1 & -1 & t-3 \end{vmatrix} \\ &= (t-3)\left((t-4)(t-3)-2\right) + \left((t-3)(-2)-2\right) - (2+t-4) \\ &= t^3 - 10t^2 + 28t - 24 \\ &= (t-2)\left(t^2 - 8t + 12\right) \\ &= (t-2)^2(t-6) \end{aligned}$$

Therefore the eigenvalues are 2 and 6.

#

For the eigenvalue 2;

$$\begin{pmatrix} 2-3 & -1 & -1 \\ -2 & 2-4 & -2 \\ -1 & -1 & 2-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -2 & -2 & -2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives $x+y+z=0$. The system has thus two free variables, for example $\{x=1, y=0, z=-1\}$ and $\{x=1, y=-1, z=0\}$. Let $u = (1,0,-1)$ and $v = (1,-1,0)$. Then all linear combinations of u and v forms a set of all possible eigenvectors, which together with zero vector comprise the eigenspace corresponding to the eigenvalue 2. In other words, u and v form a basis of the eigenspace of the eigenvalue 2.

#

For the eigenvalue 6;

$$\begin{pmatrix} 6-3 & -1 & -1 \\ -2 & 6-4 & -2 \\ -1 & -1 & 6-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 & -1 & -1 \\ -2 & 2 & -2 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Use Gaussian elimination to find the solution of this system of equation. First form an augmented matrix

$$\begin{pmatrix} 3 & -1 & -1 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix}$$

$(I) \leftrightarrow (III)$, $(3 \quad -1 \quad -1 \quad 0) \leftrightarrow (-1 \quad -1 \quad 3 \quad 0)$;

$$\begin{pmatrix} -1 & -2 & 3 & 0 \\ -2 & 2 & -2 & 0 \\ 3 & -1 & -1 & 0 \end{pmatrix}$$

$-1(I)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ -2 & 2 & -2 & 0 \\ -1 & -1 & 3 & 0 \end{pmatrix}$$

$(II) + 2(I)$, $(-2 \quad 2 \quad -2 \quad 0) + 2(1 \quad 1 \quad -3 \quad 0)$; $(III) - 3(I)$, $(3 \quad -1 \quad -1 \quad 0) - 3(1 \quad 1 \quad -3 \quad 0)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & 4 & -8 & 0 \\ 0 & -4 & -8 & 0 \end{pmatrix}$$

$\frac{1}{4}(II)$, $(1 \quad -2 \quad 0)$; $(III) + 4(II)$, $(-4 \quad -8 \quad 0) + 4(1 \quad -2 \quad 0)$;

$$\begin{pmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore rank of A is 2. The system has only one free variable, that is to say, one independent solution. Any particular non-zero solution generates its solution space, that is the eigenspace, for example $x = 1$, $y = 2$ and $z = 1$. So $w = (1, 2, 1)$ forms a basis of the eigenspace of the eigenvalue 6.

#

Let P be $\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$ Since $\det P = 1(-1) - 1(2+1) = -4 \neq 0$, hence P has an inverse. Therefore P is the required matrix such that $P^{-1}AP$ is diagonal.

#

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Quiz 2
Business Mathematics

7^{th} February 2006

Time: 1 hour (10–11pm)

1. Evaluate the following integrals. [10]

$$\int_0^1 \pi dx, \int_0^2 \left(x^2 - 3\right) dx, \int_{-1}^1 \frac{1}{z} dz, \int_0^2 \left(x^2 + \sqrt{x}\right) dx, \int_0^1 x e^x \, dx$$

Solution.

$$\int_0^1 dx = \pi(x|_0^1 = \pi$$

#

$$\int_0^2 \left(x^2 - 3\right) dx = \left(\frac{x^3}{3} - 3x\right|_0^2 = \frac{8}{3} - 6 = -\frac{10}{3}$$

#

In $\int_{-1}^1 \frac{1}{z} dz$ the limits of integration pass through 0, where there is discontinuity and where $1/z$ is undefined.

Further, $\lim_{z \to 0^-} f(z) = -\infty$ and $\lim_{z \to 0^+} f(z) = \infty$. We look at

$$\int_0^1 \frac{1}{z} dz = \lim_{a \to 0^+} (\ln z|_a^1 = \lim_{a \to 0^+} (\ln 1 - \ln a) = 0 - (-\infty) = \infty$$

and

$$\int_{-1}^1 \frac{1}{z} dz = \int_{-1}^0 \frac{1}{z} dz + \int_0^1 \frac{1}{z} dz$$

where

$$\begin{aligned}\int_{-1}^0 \frac{1}{z} dz = \lim_{b \to 0^-} \int_{-1}^b \frac{1}{(-z)} d(-z) = \lim_{b \to 0^-} (\ln(-z)|_{-1}^b &= \lim_{b \to 0^-} (\ln b - \ln(1)) \\ &= \lim_{b \to 0^-} (\ln b)\end{aligned}$$

the latter of which is undefined. Therefore $\int_{-1}^1 \frac{1}{z} dz$ is also undefined.

#

$$\int_0^2 \left(x^2 + \sqrt{x}\right) dx = \left(\frac{x^3}{3} + \frac{2}{3} x^{3/2}\right|_0^2 = \frac{8}{3} + \frac{2\sqrt{8}}{3} = \frac{8 + 4\sqrt{2}}{3}$$

#

Let $u = x$ and $dv = e^x dx$. Then $du = dx$ and $v = e^x$. And then,

$$\int_0^1 x e^x \, dx = (x e^x|_0^1 - \int_0^1 e^x \, dx = (x e^x - e^x|_0^1 = 1$$

#

Quiz 3
Business Mathematics

14^{th} February 2006

Time: 1 hour (10–11pm)

Choose only one problem[10], either

1. Solve

$$\begin{aligned} y + 3z &= 9 \\ 2x + 2y - z &= 8 \\ -x + 5z &= 8 \end{aligned}$$

by Gaussian elimination.

Solution. Write an augmented matrix,

$$\begin{bmatrix} 0 & 1 & 3 & 9 \\ 2 & 2 & -1 & 8 \\ -1 & 0 & 5 & 8 \end{bmatrix}$$

$(I) \leftrightarrow (III)$, $(0 \quad 1 \quad 3 \quad 9) \leftrightarrow (-1 \quad 0 \quad 5 \quad 8)$;

$$\begin{bmatrix} -1 & 0 & 5 & 8 \\ 2 & 2 & -1 & 8 \\ 0 & 1 & 3 & 9 \end{bmatrix}$$

$-1(I)$, $-1(-1 \quad 0 \quad 5 \quad 8)$; $(II) - 2(I)$, $(2 \quad 2 \quad -1 \quad 8) - 2(1 \quad 0 \quad -5 \quad -8)$;

$$\begin{bmatrix} 1 & 0 & -5 & -8 \\ 0 & 2 & 9 & 24 \\ 0 & 1 & 3 & 9 \end{bmatrix}$$

$II \leftrightarrow III$, $(0 \quad 2 \quad 9 \quad 24) \leftrightarrow (0 \quad 1 \quad 3 \quad 9)$; $(III) - 2(II)$, $(2 \quad 9 \quad 24) - 2(1 \quad 3 \quad 9)$;

$$\begin{bmatrix} 1 & 0 & -5 & -8 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

Therefore, directly we have $z = 2$, $y = 9 - 3(2) = 3$ and $x = -8 + 5(2) = 2$.

#

or

2. Solve

$$\begin{aligned} 2x + y - 2z &= 10 \\ 3x + 2y + 2z &= 1 \\ 5x + 4y + 3z &= 4 \end{aligned}$$

by any method.

Solution. Form an augmented matrix,

$$\begin{bmatrix} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{bmatrix}$$

$\frac{1}{2}(I)$, $(1 \quad \frac{1}{2} \quad -1 \quad 5)$; $(II)-3(I)$, $(3 \quad 2 \quad 2 \quad 1)-3(1 \quad \frac{1}{2} \quad -1 \quad 5)$; $(III)-5(I)$, $(5 \quad 4 \quad 3 \quad 4)-5(1 \quad \frac{1}{2} \quad -1 \quad 5)$;

$$\begin{bmatrix} 1 & \frac{1}{2} & -1 & 5 \\ 0 & \frac{1}{2} & 5 & -14 \\ 0 & \frac{3}{2} & 8 & -21 \end{bmatrix}$$

$2(II)$, $(1 \quad 10 \quad -28)$; $(III)-\frac{3}{2}(II)$, $\left(\frac{3}{2} \quad 8 \quad -21\right)-\frac{3}{2}(1 \quad 10 \quad -28)$;

$$\begin{bmatrix} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -7 & 21 \end{bmatrix}$$

Directly, $z=-\frac{21}{3}=-3$, $y=-28-10(-3)=2$ and $x=5-\frac{1}{2}(2)+(-3)=1$. #

Midterm examination
Business mathematics

20^{th} December 2005

Time: 3 hours (1–4pm)

1.

a. Given

$$f(x) = ax^2 + bx + c$$

where $a = c = 1$ and $b = 2$. Find all the x-intercepts, y-intercepts, and critical points.

b. Given

$$\ln y = 1 - x$$

Find y.

c. Given

$$\left(\frac{a^x a^y}{a^z}\right)^n = a^b$$

Find b.

d. Given

$$\log y = 3\log a + \log b - \log c$$

Find y.

[10]

Solution.

a.

$$f(x) = x^2 + 2x + 1$$

#

x-intercept, $y = 0$;

$$x^2 + 2x + 1 = 0$$
$$(x+1)(x+1) = 0$$
$$x = -1$$

$f(x)$ touches x-axis at $x = -1$

#

Critical point; $f'(x) = 0$;

$$2x + 2 = 0$$
$$x = -1$$

#

y-intercept, $x = 0$;

$$y = 1$$

#

b.

$$y = e^{1-x}$$

#

c.

$$\begin{aligned} a^{(x+y-z)n} &= a^b \\ b &= (x+y-z)n \end{aligned}$$

#

d.

$$\begin{aligned} \log a^3 + \log b - \log c &= \log y \\ \log \frac{a^3 b}{c} &= \log y \\ y &= \frac{a^3 b}{c} \end{aligned}$$

#

2.

a.

$$q = \ln x + \ln y$$

b.

$$z = x^3 + x^2 + x + 2xy + xy^2$$

c.

$$p = 150e^{0.74t}$$

Find the first- and second-order partial derivatives.

[10]

Solution.

a.

$$\frac{\partial q}{\partial x} = \frac{1}{x}$$

#

$$\frac{\partial q}{\partial y} = \frac{1}{y}$$

#

$$\frac{\partial^2 q}{\partial x^2} = -\frac{1}{x^2}$$

#

$$\frac{\partial^2 q}{\partial x \partial y} = 0$$

#

$$\frac{\partial^2 q}{\partial y^2} = -\frac{1}{y^2}$$

#

$$\frac{\partial^2 q}{\partial y \partial x} = 0$$

#

b.

$$\frac{\partial z}{\partial x} = 3x^2 + 2x + 1 + 2y + y^2$$

#

$$\frac{\partial z}{\partial y} = 2x + 2xy$$

#

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2$$

#

$$\frac{\partial^2 z}{\partial x \partial y} = 2 + 2y$$

#

$$\frac{\partial^2 z}{\partial y^2} = 2x$$

#

$$\frac{\partial^2 z}{\partial y \partial x} = 2 + 2y$$

#

c.

$$\frac{\partial p}{\partial t} = 150(0.74)e^{0.74t} = 111e^{0.74t}$$

#

$$\frac{\partial^2 p}{\partial t^2} = 111(0.74)e^{0.74t} = 82.14e^{0.74t}$$

#

3. Find the determinant of the following matrices.

a.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

[2]

b.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

[3]

c.

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

[5]

Solution.

a.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

#

b.

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} c & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ci - fh) - b(di - fg) + c(dh - eg) \\ &= aci - afh - bdi + bfg + cdh - ceg \end{aligned}$$

#

c.

$$
\begin{aligned}
\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix}
&= a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} \\
&\quad + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix} \\
&= a\,(f(kp - lo) - g(jp - ln) + h(jo - kn)) \\
&\quad - b\,(e(kp - lo) - g(ip - lm) + h(io - km)) \\
&\quad + c\,(e(jp - ln) - f(ip - lm) + h(in - jm)) \\
&\quad - d\,(e(jo - kn) - f(io - km) + g(in - jm)) \\
&= afkp - aflo - agjp + agln + ahjo - ahkn \\
&\quad - bekp + belo + bgip - bglm - bhio + bhkm \\
&\quad + cejp - celn - cfip + cflm + chin - chjm \\
&\quad - dejo + dekn + dfio - dfkm - dgin + dgjm
\end{aligned}
$$

#

4. The relationship between the total revenue r_t, the price p, and the output quantity q is

$$r_t = pq$$

The demand function is $p = a - bq$, where a and b are positive constants.

Find r_t, the marginal revenue r_m, and the average revenue r_a. Then find q at the maximum r_t. And then sketch the graphs of r_t, r_m and r_a. (assume $a_2 > 4b$)

[10]

Solution.

$$r_t = pq = (a - bq)q = aq - bq^2$$

#

$$r_m = \frac{dr_t}{dq} = a - 2bq$$

#

$$r_a = \frac{r_t}{q} = p = a - bq$$

#

At maximum r_t;

$$\begin{aligned} r_t' = a - 2bq &= 0 \\ q &= \frac{a}{2b} \end{aligned}$$

#

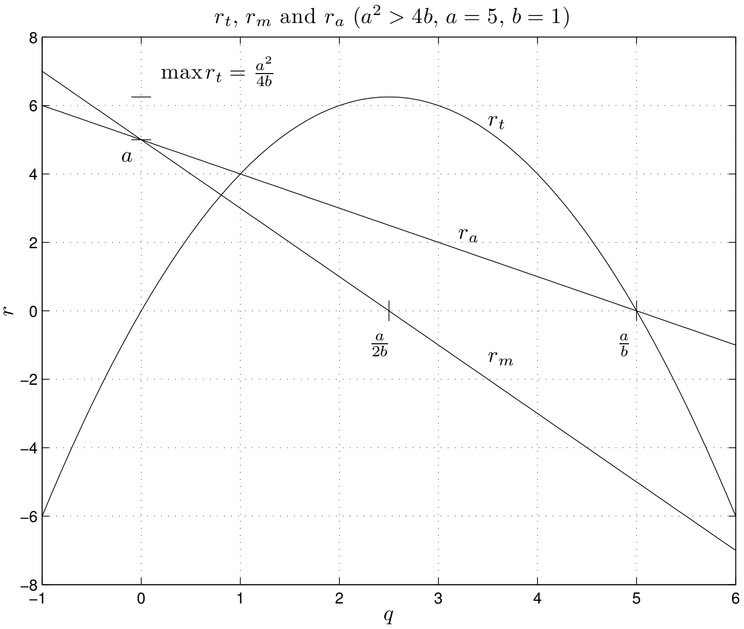


Figure 14

$$q = \frac{a}{2b}; \quad r_t = a\left(\frac{a}{2b}\right) - b\left(\frac{a^2}{4b^2}\right) = \frac{a^2}{4b^2}\left(1 - \frac{b}{2b}\right) = \frac{a^2}{4b}$$

$$q = 0; \quad r_t = 0$$

$r_t = 0;$

$$q(a - bq) = 0$$

$$q = 0, \quad \frac{a}{b}$$

$$q = 0; \quad r_m = a$$

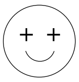
$$r_m = 0; \quad q = \frac{a}{2b}$$

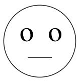
$$q = 0; \quad r_a = a$$

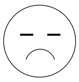
$$r_a = 0; \quad q = \frac{a}{b}$$

This can be summarised as Figure 14.

5.

a. Let A 

, B 

, and C 

represent the second-order conditions of critical point of function. Suppose the graph of a function has the shape as shown in Figure 15.

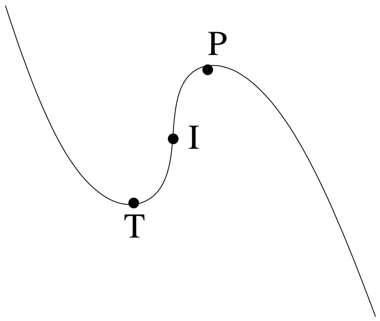


Figure 15

Which of these conditions is satisfied at points I, P and T?[3] Explain.[2]

b.

$$f(x) = \frac{3}{5}x^5 - \frac{9}{4}x^4 + x^3 + \frac{9}{2}x^2 - 6x + 7$$

Find $f'(x)$ and $f''(x)$.[1] Show that 1, -1 and 2 are the critical points.[2] Which of these are maximum, minimum or inflection point?[2]

Solution.

a.

$$\left.\begin{array}{ll} I & \leftarrow B \\ P & \leftarrow C \\ T & \leftarrow A \end{array}\right\}$$

#

At I, P and T, $f'(\cdot) = 0$.

#

At I, P and T, $f''(\cdot) = 0, \quad < 0, \quad$ and $\quad > 0$ respectively.

#

b.

$$\left.\begin{array}{ll} f'(x) & = 3x^4 - 9x^3 + 3x^2 + 9x - 6 \\ f''(x) & = 12x^3 - 27x^2 + 6x + 9 \end{array}\right\}$$

#

$$f'(x) = \left(3x^2 - 6x + 3\right)(x+1)(x-2) = 3(x-1)^2(x+1)(x-2)$$

Critical points, 1, -1 and 2. At these points $f'(\cdot) = 0$.

#

$$\left.\begin{array}{lll} f''(\cdot) = 0 & \rightarrow 1 & \text{inflection point} \\ f''(-1) < 0 & \rightarrow -1 & \text{maximum point} \\ f''(2) > 0 & \rightarrow 2 & \text{minimum point} \end{array}\right\}$$

#

6. Solve the following programme by the simplex method.

$$\begin{array}{rl} \text{maximise:} & z = 3x_1 + 4x_2 + 5x_3 \\ \text{subject to:} & x_1 + x_2 + x_3 \leq 2 \\ & x_1 + x_2 + 3x_3 \leq 1 \\ & 3x_1 + 2x_2 + x_3 \leq 4 \\ \text{with:} & \text{all the variables non-negative} \end{array}$$

[10]

Solution. Draw simplex tables.

		x_1	x_2	x_3	x_4	x_5	x_6	
		3	4	5	0	0	0	
x_4	0	1	1	1	1	0	0	2
x_5	0	1	1	3	0	1	0	1
x_6	0	3	2	1	0	0	1	4
		-3	-4	-5	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$\frac{2}{3}$	$\frac{2}{3}$	0	1	$-\frac{1}{3}$	0	$\frac{5}{3}$
x_3	$\frac{1}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	0	$\frac{1}{3}$
x_6	$\frac{8}{3}$	$\frac{5}{3}$	0	0	$-\frac{1}{3}$	1	$\frac{11}{3}$
	$-\frac{4}{3}$	$-\frac{7}{3}$	0	0	$\frac{5}{3}$	0	$\frac{5}{3}$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	-2	1	-1	0	1
x_2	1	1	3	0	1	0	1
x_6	1	0	-5	0	$\frac{4}{3}$	1	2
	1	0	7	0	4	0	4

$x_4^* = 1$, $x_2^* = 1$, $x_6^* = 2$, $x_1^* = x_3^* = x_5^* = 0$ and $z^* = 4$.

#

Final Examination
Business Mathematics

20^{th} February 2006

Time: 3 hours (9–12am)

1.

a. Give the definition of a homogeneous function. [2]

b. What are *order* and *degree* of a differential equation? [2] Given the differential equation

$$y^{(iv)} + 4\left(y''\right)^2 + \left(y'\right)^3 = \sin x$$

Give its degree and order. [2] Is this equation homogeneous? [1]

c. A *primitive* gives rise to a *differential equation.* Explain how the primitive $y = Ax^2 + Bx + C$ gives rise to the differential equation $\frac{d^3x}{dx^3} = 0$. [3]

Solution.

a. A function $f(x, y)$ is said to be *homogeneous* of degree n if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$.

#

b. The order of a differential equation is the order of its highest derivative.

#

The degree of a differential equation is the degree of its highest ordered derivative.

#

The degree is 1, and the order 4.

#

The equation is not homogeneous.

#

c.

$$\begin{aligned}\frac{dy}{dx} &= 2Ax + B\\ \frac{d^2y}{dx^2} &= 2A\\ \frac{d^3y}{dx^3} &= 0\end{aligned}$$

#

2.

a. Explain what *sequence* and *series* are. When are they said to be *arithmetic*?, when *geometric*? [6]

b. Given an arithmetic series, $1 + 3 + 5 + 7 + \cdots$. What is the value of the n^{th} term? Find the value of the 50^{th} term. What is the sum of the first n term? Find the sum of the first 50 terms. [4]

c. Given a geometric series, $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. What is the value of the n^{th} term? Find the value of the 50^{th} term. What is the sum of the first n terms? Find the sum of the first 50 terms. What is the sum to infinity, that is to say, S_∞? [5]

Solution.

a. A sequence is a list of number following a definite pattern. A series is the sum of the terms of sequence. They are said to be arithmetic if each of their terms is obtained from the term immediately preceding it by an addition of some constant. They are said to be geometric if each of their terms is obtained from the previous term by a multiplication of some constant.

#

b. From what is given, $a = 1$ and $d = 2$. The value of the n^{th} term is then $T_n = a+(n-1)d = 1+(n-1)2d$, while that of the 50^{th} term $T_{50} = 1+(50-1)2 = 99$ The sum of the first n terms is $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(2 + (n-1)2)$, and the sum of the first 50 terms is $S_{50} = \frac{50}{2}(2(1) + (50-1)2) = 2500$.

#

c. From the given geometric series, $a = 1$ and $r = \frac{1}{2}$. Then $T_n = ar^{n-1} = \left(\frac{1}{2}\right)^{n-1}$, $T_{50} = \left(\frac{1}{2}\right)^{50-1} = \frac{1}{2^{49}}$, $S_n = \frac{a(1-r^n)}{1-r} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right)$, $S_{50} = 2\left(1 - \frac{1}{2^{50}}\right) = \frac{2^{50}-1}{2^{49}}$ and $S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

#

3.

a. Give the formula for finding the present value of compound interest. Let the annual interest rate be 10 per cent. What is the present value of 1,000 Bahts in ten years' time? [4]

b. What is the present value at the end of n conversion periods in t years? Here m is the number of conversion periods per year. Find the present value of 1,000 Bahts in five years' time, when $m = 5$ and $i = 10$ per cent. [4]

c. Compare the present values obtained from (a) and (b), and discuss the difference between them. [2]

Solution.

a. The formula for the present value of compound interest is $p_t = p_0(1+i)^t$. From what is given, $i = 0.1$, $p_0 = 1000$ and $t = 5$. Therefore, $p_5 = 1000(1+0.1)^5 = 1610.51$.

#

b. The present value of compound interest at the end of $n = mt$ conversion period is $p_t = p_0\left(1 + \frac{t}{m}\right)^{mt}$. From what is given, $i = 0.1$, $m = 5$, $p_0 = 1000$ and $t = 5$. Therefore $p_5 = 1000\left(1 + \frac{0.1}{5}\right) = 1485.95$ Bahts.

#

c. $p_t > p_{t,m}$

#

4.

a. What is the *average* or *mean value* of an integrable function $f(x)$ on $[a, b]$? [2]

b. What is an integrable function? [2] Give an example of a function that is not integrable on some range. [2]

c. Consider a (2×2) system of linear equations in the slope-intercept form,

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

Give conditions for the system to have a unique solution, no solutions, and infinitely many solutions. [4]

Solution.

a. The average or mean value is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

#

b. An integrable function is a function whose value never becomes infinitely large on the range of interest.

#

The function $f(x) = \frac{1}{x}$ is not integrable over the range $[-1, 1]$.

#

c. The system has a unique solution when $m_1 \neq m_2$, no solutions when $b_1 \neq b_2$, and an infinite number of solutions when $m_1 = m_2$ and $b_1 = b_2$.

#

5.

a. Explain *revenue* and *elasticity* mathematically and verbally. [1] Explain *marginal* and *average costs* both mathematically and also in words. [1]

b. Give the condition for $a^0 = 1$ to be true. [1]

c. The following is a formal definition of limit.

> For a function $\mathrm{f}(x)$, $\lim_{n\to a} \mathrm{f}(x) = l$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathrm{f}(x) - l| < \epsilon$ whenever $0 < |x - a| < \delta$.

Explain in words what we mean by this. Give your answer both in English and in Thai. [1]

d. What is a *cut algorithm*? [1] In the following solved example, identify all pivot elements and explain in detail how the problem was solved. [5]

$$\begin{aligned} \text{maximise:}\quad & z = 2x_1 + x_2 \\ \text{subject to:}\quad & 2x_1 + 5x_2 \leq 17 \\ & 3x_1 + 2x_2 \leq 10 \\ \text{with:}\quad & x_1, x_2 \text{ non-negative and integral} \end{aligned}$$

Use cut algorithm.

Solution

Find the first approximation of Programme 1 normally using the simplex method.

		x_1	x_2	x_3	x_4	
		2	1	0	0	
x_3	0	2	5	1	0	17
x_4	0	3	2	0	1	10
		-2	-1	0	0	0

Since $\frac{10}{3} < \frac{17}{2}$, we know that 3 is the pivot element, and therefore we replace the basic variable x_4 with x_1.

	x_1	x_2	x_3	x_4	
x_3	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	$\frac{31}{3}$
x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{10}{3}$
	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{20}{3}$

We have $x_1^* = \frac{10}{3}$, $x_3^* = \frac{31}{3}$, $x_2^* = x_4^* = 0$ and $z^* = \frac{20}{3}$. Since both x_1^* and x_3^* are non-integers, arbitrarily choose the former to generate a new constraint. Then our Programme 2 becomes,

$$\begin{aligned}
x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_4 &= \frac{10}{3} = 3 + \frac{1}{3} \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &= 3 - x_1 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 - \frac{1}{3} &\geq 0 \\
\frac{2}{3}x_2 + \frac{1}{3}x_4 &\geq \frac{1}{3} \\
2x_2 + x_4 &\geq 1
\end{aligned}$$

and our new programme becomes

$$\begin{aligned}
\text{maximise:} \quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 \\
\text{subject to:} \quad & \frac{11}{3}x_2 - \frac{2}{3}x_4 = \frac{31}{3} \\
& x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_4 = \frac{10}{3} \\
& 2x_2 + x_4 \geq 1 \\
\text{with:} \quad & \text{all variables non-negative and integral}
\end{aligned}$$

		x_1	x_2	x_3	x_4	x_5	x_6	
		2	1	0	0	0	$-M$	
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	0	0	$\frac{10}{3}$
x_3	0	0	$\frac{11}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{31}{3}$
x_6	$-M$	0	2	0	1	-1	1	1
		-2	-1	0	0	0	0	0
		0	-2	0	-1	1	-1	-1

Now x_2 replaces x_6 in the basic variables and becomes the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	3
x_3	0	0	1	$-\frac{15}{6}$	$\frac{11}{6}$	$-\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	-2	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	0	0	0	0

This becomes,

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	0	$\frac{1}{3}$	3
x_3	0	0	1	$-\frac{5}{2}$	$\frac{11}{6}$	$\frac{17}{2}$
x_2	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
	0	0	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{13}{2}$

Then our first approximation of Programme 2 is $x_1^* = 3$, $x_2^* = \frac{1}{2}$, $x_3^* = \frac{17}{2}$, $x_4^* = x_5^* = 0$, and $z^* = \frac{13}{2}$. Arbitrarily choose x_2^* to generate the new constraint.

$$\begin{aligned}
x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 &= \frac{1}{2} \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &= -x_2 \\
\frac{1}{2}x_4 - \frac{1}{2}x_5 - \frac{1}{2} &\geq 1 \\
x_4 - x_5 &\geq 1
\end{aligned}$$

Then our Programme 3 becomes,

$$\begin{aligned}
\text{maximise:}\quad & z = 2x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\
\text{subject to:}\quad & x_1 + \frac{1}{3}x_5 = 3 \\
& x_3 - \frac{5}{2}x_4 + \frac{11}{6}x_5 = \frac{17}{2} \\
& x_2 + \frac{1}{2}x_4 - \frac{1}{2}x_5 = \frac{1}{2} \\
& x_4 - x_5 \geq 1 \\
\text{with:}\quad & \text{all variables non-negative and integral}
\end{aligned}$$

We draw our table for this programme.

		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
		2	1	0	0	0	0	$-M$	
x_1	0	1	0	0	0	$\frac{1}{3}$	0	0	3
x_2	0	0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$
x_3	0	1	0	0	$-\frac{5}{2}$	$\frac{11}{6}$	0	0	$\frac{17}{2}$
x_7	$-M$	0	0	0	1	-1	-1	1	1
		-2	-1	0	0	0	0	0	0
		0	0	0	-1	1	1	-1	-1

Then x_4 replaces the basic x_7 to become the pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	1	0	0	0	$-\frac{2}{3}$	$-\frac{5}{2}$	11
x_4	0	0	0	1	-1	-1	1
	-2	-1	0	0	0	0	0

Next, x_1 remains basic and becomes a pivot element.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	-1	0	0	$\frac{1}{3}$	0	6

This becomes

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	0	0	$\frac{1}{3}$	0	3
x_2	0	1	0	0	0	$\frac{1}{2}$	0
x_3	0	0	0	0	-1	$-\frac{5}{2}$	8
x_4	0	0	0	1	-1	-1	1
	0	0	0	0	$\frac{1}{3}$	$\frac{1}{2}$	6

The optimum point for Programme 3 is then, $x_1^* = 3$, $x_3^* = 8$, $x_4^* = 1$, $x_2^* = x_5^* = x_6^* = 0$ and $z^* = 6$. Therefore the solution to the original problem Programme 1 is $x_1^* = 3$, $x_2^* = 0$ at the objective value $z^* = 6$.

#

Students' scores
Business Mathematics
2005–6

Kit Tyabandha, PhD

14^{th} January, 2007

Introduction

This is a report of study and examination results for the course Business Mathematics taught by me during the second semester, 2005–6 academic year. It has been my concern from the beginning of 2006 that students had not done their homework themselves, therefore I have had since then practices and quizzes in class, with the hope that they would obtain some problem-solving skill. Everything, which includes exams, quizzes and practices, was done in an opened-book manner because that is how the real world works and that is where the progress lies.

Midterm exam

There were six questions in the midterm exam, namely from M1 to M6. MT is the total marks and MM the actual contribution toward the students' final score. Table 6 shows the six marks, $M1$–$M6$, for each student ID, together with the total mark and the same in per cent for each.

ID	*M1*	*M2*	*M3*	*M4*	*M5*	*M6*	*Total* (60)	*Per cent* (20)
4661	0.5	8.3	5.2	0	2.2	1.5	17.7	5.9
4801	6	7.5	5.2	0	2.7	3	24.4	8.1
4802	7	6.4	5.2	0	2	3	23.6	7.9
4803	6.5	7.6	5.2	3.5	3	1.5	27.3	9.1
4804								
4805	0	5	5.2	0	0	4.7	14.9	5
4806	7	3.3	5.2	0	1	2	18.5	6.2
4807	7	0.8	5.2	1	1	0	15	5
4808	1	4.2	5.2	1	1	0	12.4	4.1
4809	1	5	5.2	1	1	2	15.2	5.1
4810	1.5	5	5.2	3.5	1	0	16.2	5.4

Table 6 *Midterm marks, Business Mathematics, 2005–6, contribution of 20 per cent toward the overall points*

ID	*M1*	*M2*	*M3*	*M4*	*M5*	*M6*	*Total* (60)	*Per cent* (20)
4811	6	7.6	5.2	0	1	0	19.8	6.6
4812								
4813	6.5	7.6	5.2	0	2	0	21.3	7.1
4814	6	9	5.2	3.5	3	1.5		
4815	8	8.7	9	3.5	1.7	1.5	32.4	10.8
4816	5	6.8	8	3.5	2	0	25.3	8.4
4817	1	1.7	5.2	1	0	2	10.9	3.6
4818	4.5	10	5.2	1	3	1	24.7	8.2
4819	1	5	5.2	1	2.1	1.5	16.8	5.3
4820	0.5	0	5.2	1	1.9	1.5	10.1	3.4
4821	1	2.1	5.2	1	2.1	1.5	12.9	4.3
4822	2	2.1	5.2	0	2	1.5	12.8	4.3
4823	6.5	7.6	9	3.5	2	1.5	30.1	10
4824	7.5	7.6	9	3.5	2	0	29.6	9.9
4825	5	8.3	5.1	0	0	1.7	20.1	6.7
4826	6.5	8.8	9	3.5	3	1.5	32.3	10.8
4827	7	8.3	9	3.5	3	1.5	3.2	10.8
4828								
4829	7.5	8.2	9	3.5	2	0	30.2	10.1
4830	3	2.1	5.2	0	1	0	11.3	3.8
4831	2	2.1	5.2	1	2.1	1.5	13.9	4.6
4832	1	1.8	5.2	1	1.5	1.5	12	4
4833	0.5	1.8	5.2	2	2	1.5	13	4.3
4834								
4835	0.5	2.1	5.2	2	1.5	1.5	12.8	4.3
4836	2	1.8	5.2	2	2	1.5	14.5	4.8
4837	6.5	7.6	9	3.5	2.3	1.5	30.4	10.1
4838	6.5	7.6	9	3.5	3	1.5	31.1	10.4
4839	6	8.3	9	3.5	3.3	1.5	31.6	10.5
4840	8.5	7.6	9	3.5	3	1.5	33.1	11
4841	8.5	7.6	9	8.5	3	1.5	38.1	12.7
4842	8	7.9	5.2	2	2	1.5	26.6	8.9
4843	8	7.6	9	8.5	3	1.5	37.6	12.5
4844	0.5	2.5	5	0	2	1.5	11.5	3.8
4845	0.5	8.3	5	0	2	1.5	17.3	5.8
4846	0.5	8.3	5.2	0.5	2.7	1.5	18.7	6.2
4847	8	8.3	9	3.5	2	1.5	32.3	10.8
4848	8	7	9	8.5	2	1.5	36	12
4849								
4850	7.5	7.9	9	3.5	2	1.5	31.4	10.5
4851	6	6.6	5.2	3.5	2.8	1.5	25.6	8.5
4852	6	6.6	5.2	3.5	2.5	1.5	25.3	8.4

Table 6 *(continued) Midterm marks.*

ID	*M1*	*M2*	*M3*	*M4*	*M5*	*M6*	*Total* (60)	*Per cent* (20)
4853								
4854	6	3.9	5.2	3	3	1.5	22.6	7.5
4855	8.5	7.5	5.2	3	3	1.2	28.4	9.5
4856	8	7.5	7	0.5	0.5	1.5	25	8.3
4857	7	8.3	8	0.5	2.7	1.5	28	9.3
4858	8.5	7.9	8	1.5	2.7	1.5	30.1	10
4859	8	8.3	7.8	0.5	2.7	1.5	28.8	9.6
4860								
4861	5	8.3	8	0	2	3	26.3	8.8
4862	8.5	2.9	5.2	3	2.5	1.5	23.6	7.9
4863	8	2	5.2	3.5	2.5	1.5	22.7	7.6
4864								
4865	6.5	6.6	5.2	3.5	3	1.5	26.3	8.8
4866	6	5.1	5.2	3.5	3	1.5	24.3	8.1
4867	7	2.1	5.2	3	3	1.5	21.8	7.3
4868	7	2.2	5.2	3	2.5	1.5	21.4	7.1
4869								
4870	8	7.9	5.2	3	0.7	1.5	26.3	8.8
4871	6	7.6	8	3	5	1.5	31.1	10.4
4872	7	7	8	3	2.7	1.5	29.2	9.7

Table 6 *(continued) Midterm marks.*

Table 7 shows mark and rank of each student.

ID	*score*	*rank*	*ID*	*score*	*rank*	*ID*	*score*	*rank*	*ID*	*score*	*rank*
4661	5.9	38	4818	8.23	25	4836	4.83	45	4854	7.53	30
4801	8.13	26	4819	5.27	41	4837	10.13	10	4855	9.47	16
4802	7.87	28	4820	3.37	55	4838	10.37	9	4856	8.33	24
4803	9.1	19	4821	4.3	48	4839	10.53	7	4857	9.33	18
4805	4.97	44	4822	4.27	49	4840	11.03	4	4858	10.03	12
4806	6.17	37	4823	10.03	12	4841	12.7	1	4859	9.6	15
4807	5	43	4824	9.87	13	4842	8.87	20	4861	8.77	21
4808	4.13	50	4825	6.7	34	4843	12.53	2	4862	7.87	28
4809	5.07	42	4826	10.77	6	4844	3.83	52	4863	7.57	29
4810	5.4	40	4827	10.77	6	4845	5.77	39	4865	8.77	21
4811	6.6	35	4829	10.07	11	4846	6.23	36	4866	8.1	27
4813	7.1	33	4830	3.77	53	4847	10.77	6	4867	7.27	31
4814	9.4	17	4831	4.63	46	4848	12	3	4868	7.13	32
4815	10.8	5	4832	4	51	4850	10.47	8	4870	8.77	21
4816	8.43	23	4833	4.33	47	4851	8.53	22	4871	10.37	9
4817	3.63	54	4835	4.27	49	4852	8.43	23	4872	9.73	14

Table 7 *Mark and rank of students' midterm scores.*

Figure 16 shows the distribution of the scores.

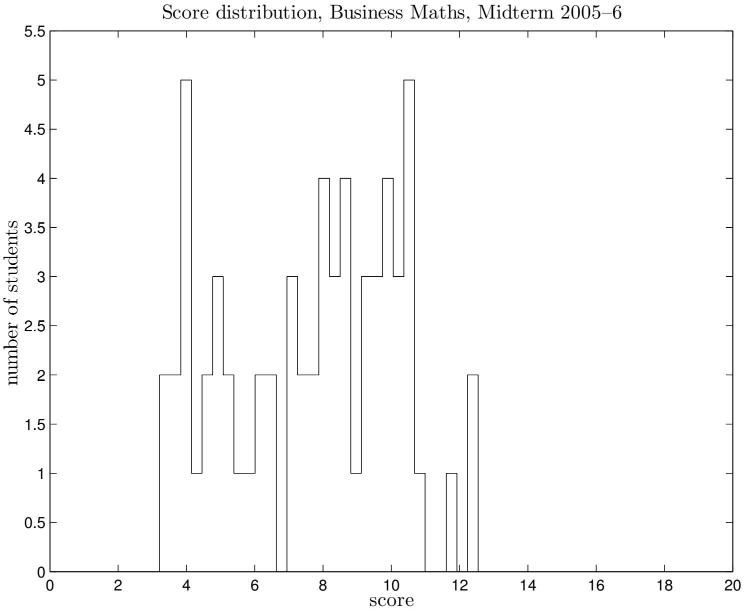


Figure 16 *Distribution of students' midterm scores.*

The mean score for the midterm exam is 7.78, median 8.18, minimum 3.37 and maximum 12.7. The standard deviation is 2.53.

Quiz I

This first one of the series of quizzes was held on 31 January 2006. The scores have as its mean 2.38, median 1.9, minimum 1 and maximum 6.3. The standard deviation is 1.18.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4801	1.6	21	4820	1	26	4838	4	7	4855	1	26
4802	2.1	16	4821	1.8	19	4839	2	17	4856	4	7
4803	4.5	2	4822	1.6	21	4840	1.6	21	4857	4.4	3
4805	1.6	21	4823	1.6	21	4841	4.1	6	4858	1.7	20
4806	2.4	13	4824	1.6	21	4842	1.9	18	4859	4	7
4807	1.6	21	4825	1.6	21	4843	2.8	10	4861	1.6	21
4808	1.4	23	4826	4.2	5	4844	2.8	10	4862	1.6	21
4809	1.5	22	4827	3.9	8	4845	1.8	19	4863	2.9	9
4810	2.4	13	4829	2.2	15	4846	1	26	4865	2.4	13
4811	4	7	4830	1	26	4847	1.2	25	4866	1.8	19
4813	4	7	4831	4	7	4848	2.1	16	4867	2.7	11
4814	1	26	4832	1.5	22	4849	1.7	20	4868	1.6	21
4815	2.4	13	4833	1.4	23	4850	2	17	4870	4	7
4817	2.3	14	4835	1.3	24	4851	2.6	12	4871	1.9	18
4818	6.3	1	4836	2.2	15	4852	1.7	20	4872	4	7
4819	1.9	18	4837	1	26	4854	4.3	4			

Figure 17 shows the score distribution of Quiz 1.

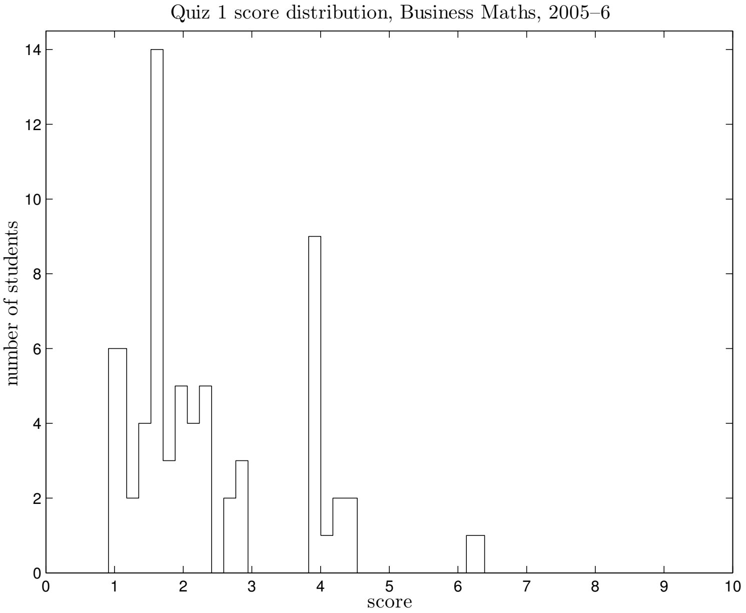


Figure 17 *Distribution of students' scores from Quiz 1.*

Quiz II

The second quiz was held on 7 February 2006.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	5.2	18	4818	8	3	4836	5.7	15	4854	8.2	2
4801	8.6	1	4819	4.2	22	4837	4	23	4855	7.8	5
4802	6.1	13	4820	4	23	4838	7.2	9	4856	8.2	2
4803	8.2	2	4821	4	23	4839	3.7	24	4857	8.2	2
4805	6	14	4822	7.9	4	4840	5.2	18	4858	7	10
4806	6	14	4823	7.7	6	4841	6.2	12	4859	8.2	2
4807	7.9	4	4824	8.2	2	4842	5.4	17	4861	6.9	11
4808	6.9	11	4825	5	20	4843	8.2	2	4862	7.6	7
4809	3.4	25	4826	8.2	2	4845	4.2	22	4863	8.2	2
4810	8.2	2	4827	8.2	2	4846	7.2	9	4865	5.1	19
4811	7.2	9	4829	8.2	2	4847	7	10	4866	4.2	22
4813	8.2	2	4830	4.2	22	4848	8.2	2	4867	5.2	18
4814	8.2	2	4831	8.2	2	4849	7.7	6	4868	7.3	8
4815	8.2	2	4832	6	14	4850	8.2	2	4870	7.7	6
4816	6.2	12	4833	4	23	4851	5.1	19	4871	4.7	21
4817	4	23	4835	7.8	5	4852	5.5	16	4872	8.2	2

Table 8 *Quiz II marks, Business Mathematics, 2005–6, contribution of 10 per cent toward the overall points*

The mean score for Quiz 2 is 6.62, median 7.2, minimum 3.4 and maximum 8.6. The standard deviation of the score is 1.62. The distribution of score is then Figure 18.

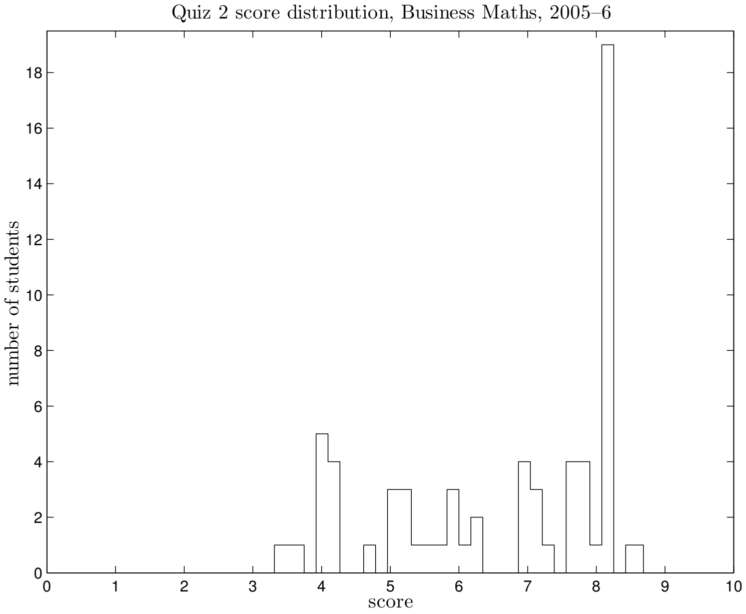


Figure 18 *Distribution of students' quiz 2's scores.*

Quiz 3

We had our Quiz 3 on 14 February 2006, Valentine Day. From the results, the mean is 8.15, median 10, minimum 2, maximum 10, and the standard deviation 3.14.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	10	1	4818	10	1	4838	2	5	4856	2	5
4801	10	1	4819	10	1	4839	2	5	4857	10	1
4802	10	1	4820	2	5	4840	10	1	4858	10	1
4803	10	1	4822	10	1	4841	10	1	4859	10	1
4805	10	1	4823	10	1	4842	5	3	4861	4	4
4806	10	1	4824	10	1	4843	10	1	4862	10	1
4807	10	1	4826	10	1	4844	9	2	4863	10	1
4808	2	5	4827	10	1	4845	2	5	4865	9	2
4809	2	5	4829	10	1	4846	10	1	4866	10	1
4810	10	1	4830	10	1	4847	4	4	4867	10	1
4811	10	1	4831	2	5	4848	10	1	4868	10	1
4813	2	5	4832	10	1	4849	5	3	4870	10	1
4814	10	1	4833	10	1	4850	10	1	4871	10	1
4815	10	1	4835	5	3	4852	10	1	4872	10	1
4816	10	1	4836	9	2	4854	10	1			
4817	2	5	4837	10	1	4855	5	3			

Table 9 *Quiz 3 marks, Business Mathematics, 2005–6, contribution of 10 per cent toward the overall points*

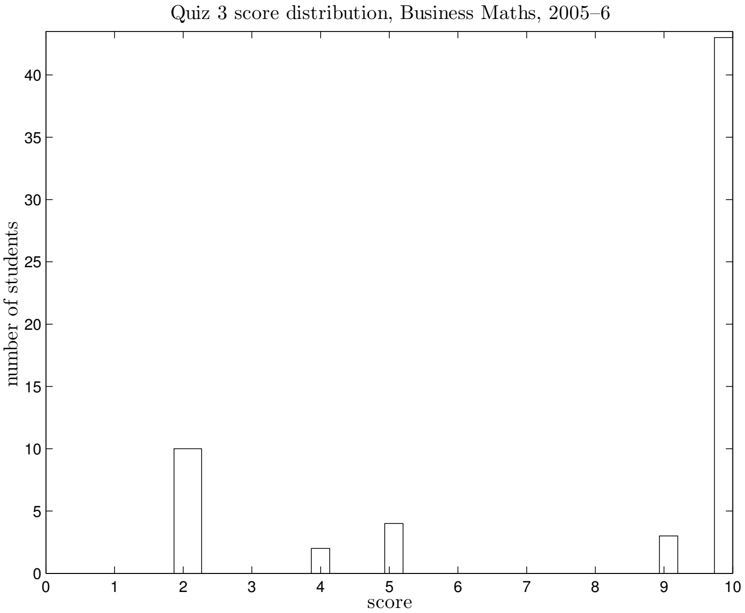


Figure 19 *Distribution of students' quiz 3's scores.*

Final Exam

The final examination was held on 20 February 2006. There were five questions, with marks respectively from the first to the last, 10, 15, 10, 10 and 10, totalling 55.

ID	*Q1*	*Q2*	*Q3*	*Q4*	*Q5*	*Total* (55)	*Per cent*	*Scaled* (30)
4661	2	4.9	3	0	0.5	10.4	18.91	5.67
4801	5	0	0	4	2.8	11.8	21.45	6.44
4802	2	6.4	2	4	3	17.4	31.64	9.49
4803	2	0	1.1	4	1	8.1	14.73	4.42
4805	4	6.5	0	6.1	4	20.6	37.45	11.24
4806	1	12.5	0	3	2	18.5	33.64	10.09
4807	5	6.2	0	2.1	5.8	19.1	34.73	10.42
4808	3	0	0	0	1.8	4.8	8.73	2.62
4809	2	6.4	0	0.1	0	8.5	15.45	4.64
4810	3	10.4	0	3.1	2	18.5	33.64	10.09
4811	0	1.7	0	4.1	1	6.8	12.36	3.71
4813	2	6	0	0	2	10	18.18	5.45
4814	5	6.5	0	4	2.8	18.3	33.27	9.98
4815	1	6.9	0	2.1	3	13	23.64	7.09
4816	3	6.7	0	0.1	0	9.8	17.82	5.35
4817	2	0	2	4.1	0.2	8.3	15.09	4.53
4818	3	7	2.2	5	3	20.2	36.73	11.02
4819	2	0	0	4	0	6	10.91	3.27
4820	0	0.2	0	0	1	1.2	2.18	0.65
4821	2	0	0	0	6.8	8.8	16	4.8
4822	0	10	3.1	4	8	25.1	45.64	13.69
4823	1	9.3	0	1	1.5	12.8	23.27	6.98
4824	5	8.3	4.1	6.1	6.3	29.8	54.18	16.25
4825	4	6	2.1	4	0	16.1	29.27	8.78
4826	2	0.2	0	6.1	3.5	11.8	21.45	6.44
4827	1	0	0	4	2	7	12.73	3.82
4829	0	10.5	0	0	2	12.5	22.73	6.82
4830	0	2	0	0	0	2	3.64	1.09
4831	0	13.8	9	1	0	23.8	43.27	12.98
4832	2	0	0	4	0	6	10.91	3.27
4833	0	6.5	2	6	2	16.5	30	9

Table 10 *Final marks (continued)*

ID	*Q1*	*Q2*	*Q3*	*Q4*	*Q5*	*Total* (55)	*Per cent*	*Scaled* (30)
4835	1	6.5	2	4	0	13.5	24.55	7.36
4836	3	0	4.1	5.1	3.3	15.5	28.18	8.45
4837	1	2	2.2	4	1	10.2	18.55	5.56
4838	0	0.2	0	0	1	1.2	2.18	0.65
4839	5	12.5	1.1	2.1	5	25.7	46.73	14.02
4840	3	12.9	4	3.1	2	25	45.45	13.64
4841	4	8.8	0	4	3.8	20.6	37.45	11.24
4842	6	13.9	3.2	4	0	27.1	49.27	14.78
4843	3	13.5	6.2	4	6.8	33.5	60.91	18.27
4844	7	1.7	0	0	1	9.7	17.64	5.29
4845	2	5.5	2.4	0.1	0	10	18.18	5.45
4846	0	0	0	0.1	0	0.1	0.18	0.05
4847	0	0.5	0	0	0	0.5	0.91	0.27
4848	1	10.1	1.1	3	3	18.2	33.09	9.93
4849	2	10.1	2.2	5.1	4.8	24.2	44	13.2
4850	3	7.3	0	4	5	19.3	35.09	10.53

Table 10 *Final examination marks, Business Mathematics, 2005–6, contributing 30 per cent towards overall points*

ID	*Q1*	*Q2*	*Q3*	*Q4*	*Q5*	*Total* (55)	*Per cent*	*Scaled* (30)
4851	2	0	0	4	0	6	10.91	3.27
4852	2	0	0	4	4	10	18.18	5.45
4854	6	0	0	6	3	15	27.27	8.18
4855	2.2	8	1	4	4	19.2	34.91	10.47
4856	6	0	4	4	0	14	25.45	7.64
4857	4.1	2.2	10	6	2	24.3	44.18	13.25
4858	0	6.8	9	6.1	2	23.9	43.45	13.04
4859	3	6.2	0	6.1	0	15.3	27.82	8.35
4860	4.5	13.5	3.2	4.1	0	25.3	46	13.8
4861	2	6.1	0	4	3	15.1	27.45	8.24
4862	2	9.9	0	4	4.8	20.7	37.64	11.29
4863	3	12.6	2.8	7	3.9	29.3	53.27	15.98
4865	4	0	0	4	2.8	10.8	19.64	5.89
4866	3	3	2	0	0	8	14.55	4.36
4867	0	0	0	4	0	4	7.27	2.18
4868	0	0.2	1	0	0	1.2	2.18	0.65
4870	0	4.2	0	0	0	4.2	7.64	2.29
4871	4	8.7	3.1	4	5	24.8	45.09	13.53
4872	1	0	0	7	0	8	14.55	4.36

Table 10 *(continued) Final exam marks.*

From this we scale the total down to 30. The mean resulted is thus 7.74, the median 7.23, minimum 0.05, maximum 18.27, and the standard deviation 4.46.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	5.67	37	4819	3.27	49	4838	0.65	54	4856	7.64	30
4801	6.44	35	4820	0.65	54	4839	14.02	5	4857	13.25	10
4802	9.49	23	4821	4.8	42	4840	13.64	8	4858	13.04	12
4803	4.42	45	4822	13.69	7	4841	11.24	15	4859	8.35	27
4805	11.24	15	4823	6.98	33	4842	14.78	4	4860	13.8	6
4806	10.09	20	4824	16.25	2	4843	18.27	1	4861	8.24	28
4807	10.42	19	4825	8.78	25	4844	5.29	41	4862	11.29	14
4808	2.62	50	4826	6.44	35	4845	5.45	39	4863	15.98	3
4809	4.64	43	4827	3.82	47	4846	0.05	56	4865	5.89	36
4810	10.09	20	4829	6.82	34	4847	0.27	55	4866	4.36	46
4811	3.71	48	4830	1.09	53	4848	9.93	22	4867	2.18	52
4813	5.45	39	4831	12.98	13	4849	13.2	11	4868	0.65	54
4814	9.98	21	4832	3.27	49	4850	10.53	17	4870	2.29	51
4815	7.09	32	4833	9	24	4851	3.27	49	4871	13.53	9
4816	5.35	40	4835	7.36	31	4852	5.45	39	4872	4.36	46
4817	4.53	44	4836	8.45	26	4854	8.18	29			
4818	11.02	16	4837	5.56	38	4855	10.47	18			

Table 11 *Final examination marks and ranks, Business Mathematics, 2005–6, totalling 30 per cent*

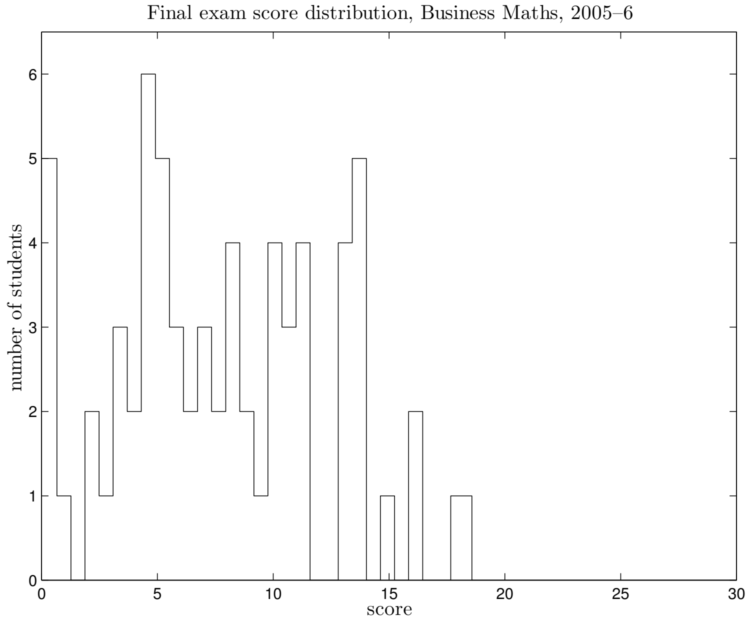


Figure 20 *Distribution of students' final examination's scores.*

Total exam score

All two exams and three quizzes total to 80 per cent. Of this the average score is at 31.63, the median 32.03, minimum 11.02, maximum 51.81 and standard deviation 8.85. Then the scores and their ranks are shown in Table 12 and the distribution of scores in Figure 21.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	26.77	47	4819	24.64	52	4838	24.22	54	4856	30.17	40
4801	34.77	28	4820	11.02	66	4839	32.25	33	4857	45.19	3
4802	35.56	26	4821	14.9	64	4840	41.47	9	4858	41.77	8
4803	36.22	23	4822	37.46	18	4841	44.24	5	4859	40.15	12
4805	33.8	30	4823	36.32	21	4842	35.95	25	4860	13.8	65
4806	34.66	29	4824	45.92	2	4843	51.81	1	4861	29.5	42
4807	34.92	27	4825	22.08	56	4844	20.92	57	4862	38.36	16
4808	17.05	61	4826	39.6	13	4845	19.22	60	4863	44.65	4
4809	16.6	62	4827	36.68	20	4846	24.49	53	4865	31.16	36
4810	36.09	24	4829	37.28	19	4847	23.24	55	4866	28.46	44
4811	31.51	35	4830	20.06	58	4848	42.23	7	4867	27.35	46
4813	26.75	48	4831	31.82	34	4849	27.6	45	4868	26.69	49
4814	38.58	14	4832	24.77	51	4850	41.19	10	4870	32.76	32
4815	38.49	15	4833	28.73	43	4851	19.51	59	4871	40.49	11
4816	29.98	41	4835	25.73	50	4852	31.09	37	4872	36.3	22
4817	16.46	63	4836	30.19	39	4854	38.22	17			
4818	43.55	6	4837	30.7	38	4855	33.74	31			

Table 12 *Total exam score, Business Mathematics, 2005–6, 80 per cent of total*

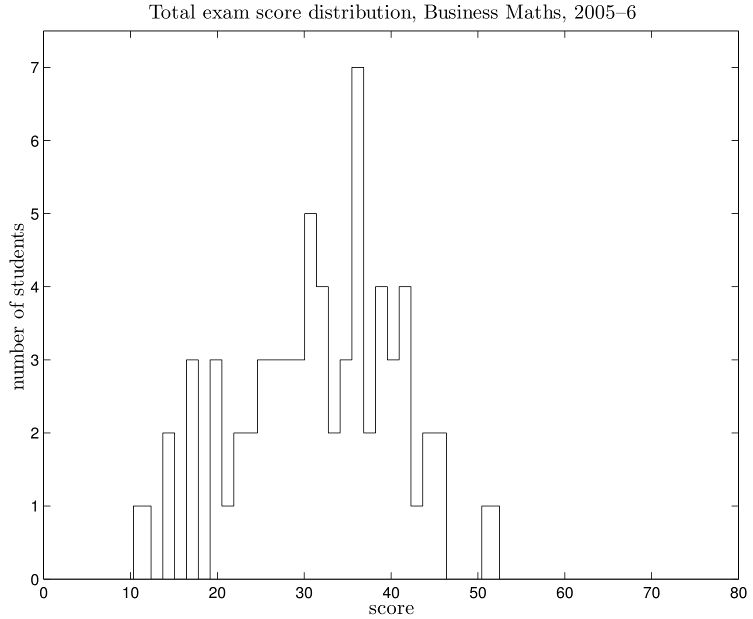


Figure 21 *Distribution of total scores from all three quizzes and two exams.*

Homework

For the marks for homeworks the mean is 8.08, median 8.00, minimum 7.00, maximum 9.00 and standard deviation 0.44.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	7	3	4819	8	2	4838	8	2	4856	8	2
4801	8	2	4820	8	2	4839	8	2	4857	8	2
4802	8	2	4821	8	2	4840	8	2	4858	9	1
4803	8	2	4822	8	2	4841	8	2	4859	9	1
4805	9	1	4823	8	2	4842	8	2	4860	7	3
4806	8	2	4824	8	2	4843	9	1	4861	9	1
4807	8	2	4825	8	2	4844	7	3	4862	8	2
4808	9	1	4826	9	1	4845	8	2	4863	8	2
4809	8	2	4827	8	2	4846	8	2	4865	8	2
4810	8	2	4829	8	2	4847	9	1	4866	8	2
4811	8	2	4830	8	2	4848	9	1	4867	8	2
4813	7	3	4831	8	2	4849	8	2	4868	8	2
4814	8	2	4832	8	2	4850	8	2	4870	8	2
4815	8	2	4833	8	2	4851	8	2	4871	8	2
4816	8	2	4835	8	2	4852	8	2	4872	8	2
4817	8	2	4836	8	2	4854	8	2			
4818	8	2	4837	8	2	4855	8	2			

Attendance

The scores for attendance have as their average 9.33, median 9.5, minimum 5.00, maximum 10.00 and standard deviation 1.02. Scores and their ranks are given in Table 13.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	5	8	4819	10	1	4838	9	3	4856	9.5	2
4801	8	5	4820	9.5	2	4839	10	1	4857	10	1
4802	9.5	2	4821	9.5	2	4840	9	3	4858	10	1
4803	9.5	2	4822	8.5	4	4841	10	1	4859	9	3
4805	10	1	4823	8.5	4	4842	10	1	4860	5	8
4806	9.5	2	4824	9.5	2	4843	10	1	4861	10	1
4807	8.5	4	4825	6.5	7	4844	9.5	2	4862	7.5	6
4808	9.5	2	4826	9.5	2	4845	10	1	4863	10	1
4809	9.5	2	4827	10	1	4846	9.5	2	4865	9	3
4810	10	1	4829	10	1	4847	10	1	4866	10	1
4811	9	3	4830	9	3	4848	10	1	4867	10	1
4813	8.5	4	4831	9	3	4849	9.5	2	4868	9.5	2
4814	9.5	2	4832	9.5	2	4850	9.5	2	4870	10	1
4815	10	1	4833	9.5	2	4851	9.5	2	4871	10	1
4816	9.5	2	4835	10	1	4852	9.5	2	4872	10	1
4817	10	1	4836	10	1	4854	10	1			
4818	10	1	4837	8.5	4	4855	9	3			

Table 13 *Score and rank of attendance.*

Total score including homework and attendance

We compile the results of quizzes, exams, homework and attendance into Table 14.

ID	*Midterm*	*Quiz 1*	*Quiz 2*	*Quiz 3*	*Final*	*HW & Atd*
4661	5.9	0	5.2	10	5.67	12
4801	8.13	1.6	8.6	10	6.44	16
4802	7.87	2.1	6.1	10	9.49	17.5
4803	9.1	4.5	8.2	10	4.42	17.5
4805	4.97	1.6	6	10	11.24	19
4806	6.17	2.4	6	10	10.09	17.5
4807	5	1.6	7.9	10	10.42	16.5
4808	4.13	1.4	6.9	2	2.62	18.5
4809	5.07	1.5	3.4	2	4.64	17.5
4810	5.4	2.4	8.2	10	10.09	18
4811	6.6	4	7.2	10	3.71	17
4813	7.1	4	8.2	2	5.45	15.5
4814	9.4	1	8.2	10	9.98	17.5
4815	10.8	2.4	8.2	10	7.09	18
4816	8.43	–	6.2	10	5.35	17.5
4817	3.63	2.3	4	2	4.53	18
4818	8.23	6.3	8	10	11.02	18
4819	5.27	1.9	4.2	10	3.27	18
4820	3.37	1	4	2	0.65	17.5
4821	4.3	1.8	4	–	4.8	17.5
4822	4.27	1.6	7.9	10	13.69	16.5
4823	10.03	1.6	7.7	10	6.98	16.5
4824	9.87	1.6	8.2	10	16.25	17.5
4825	6.7	1.6	5	–	8.78	14.5
4826	10.77	4.2	8.2	10	6.44	18.5
4827	10.77	3.9	8.2	10	3.82	18
4829	10.07	2.2	8.2	10	6.82	18
4830	3.77	1	4.2	10	1.09	17

Table 14 *All results compiled into a single table, Business Mathematics, 2005–6*

ID	*Midterm*	*Quiz 1*	*Quiz 2*	*Quiz 3*	*Final*	*HW & Atd*
4831	4.63	4	8.2	2	12.98	17
4832	4	1.5	6	10	3.27	17.5
4833	4.33	1.4	4	10	9	17.5
4835	4.27	1.3	7.8	5	7.36	18
4836	4.83	2.2	5.7	9	8.45	18
4837	10.13	1	4	10	5.56	16.5
4838	10.37	4	7.2	2	0.65	17
4839	10.53	2	3.7	2	14.02	18
4840	11.03	1.6	5.2	10	13.64	17
4841	12.7	4.1	6.2	10	11.24	18
4842	8.87	1.9	5.4	5	14.78	18
4843	12.53	2.8	8.2	10	18.27	19
4844	3.83	2.8	–	9	5.29	16.5
4845	5.77	1.8	4.2	2	5.45	18
4846	6.23	1	7.2	10	0.05	17.5
4847	10.77	1.2	7	4	0.27	19
4848	12	2.1	8.2	10	9.93	19
4849	–	1.7	7.7	5	13.2	17.5
4850	10.47	2	8.2	10	10.53	17.5
4851	8.53	2.6	5.1	–	3.27	17.5
4852	8.43	1.7	5.5	10	5.45	17.5
4854	7.53	4.3	8.2	10	8.18	18
4855	9.47	1	7.8	5	10.47	17
4856	8.33	4	8.2	2	7.64	17.5
4857	9.33	4.4	8.2	10	13.25	18
4858	10.03	1.7	7	10	13.04	19
4859	9.6	4	8.2	10	8.35	18
4860	–	–	–	–	13.8	12
4861	8.77	1.6	6.9	4	8.24	19
4862	7.87	1.6	7.6	10	11.29	15.5
4863	7.57	2.9	8.2	10	15.98	18
4865	8.77	2.4	5.1	9	5.89	17
4866	8.1	1.8	4.2	10	4.36	18
4867	7.27	2.7	5.2	10	2.18	18
4868	7.13	1.6	7.3	10	0.65	17.5
4870	8.77	4	7.7	10	2.29	18
4871	10.37	1.9	4.7	10	13.53	18
4872	9.73	4	8.2	10	4.36	18

Table 14 *(continued) All results compiled into a single table, Business Mathematics, 2005–6.*

The total score has the mean 49.04, median 49.53, minimum 25.8, maximum 70.81 and standard deviation 9.31. The score and ranking for every student are shown in Table 15.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	38.77	55	4819	42.64	49	4838	41.22	54	4856	47.67	40
4801	50.77	30	4820	28.52	65	4839	50.25	33	4857	63.19	3
4802	53.06	25	4821	32.4	64	4840	58.47	11	4858	60.77	8
4803	53.72	24	4822	53.96	21	4841	62.24	5	4859	58.15	12
4805	52.8	27	4823	52.82	26	4842	53.95	22	4860	25.8	66
4806	52.16	28	4824	63.42	2	4843	70.81	1	4861	48.5	37
4807	51.42	29	4825	36.58	60	4844	37.42	56	4862	53.86	23
4808	35.55	61	4826	58.1	13	4845	37.22	57	4863	62.65	4
4809	34.1	63	4827	54.68	18	4846	41.99	53	4865	48.16	39
4810	54.09	20	4829	55.28	17	4847	42.24	52	4866	46.46	43
4811	48.51	36	4830	37.06	58	4848	61.23	7	4867	45.35	45
4813	42.25	51	4831	48.82	34	4849	45.1	46	4868	44.19	47
4814	56.08	16	4832	42.27	50	4850	58.69	9	4870	50.76	31
4815	56.49	14	4833	46.23	44	4851	37.01	59	4871	58.49	10
4816	47.48	41	4835	43.73	48	4852	48.59	35	4872	54.3	19
4817	34.46	62	4836	48.19	38	4854	56.22	15			
4818	61.55	6	4837	47.2	42	4855	50.74	32			

Table 15 *Total score and ranking, Business Mathematics, 2005–6.*

We had some students absent from some of the exams and tests. For them we take the total marks as proportionally less than 100. Thus the dividing factors for them are namely 0.9 for students 4815, 4821, 4825, 4844 and 4851. For student 4849 it is 0.8, and for 4860, 0.5. The mean then becomes 49.93, the median 50.76, minimum 28.52, maximum 70.81 and standard deviation 8.55. The score and ranking become that given in Table 16.

ID	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*	*ID*	*Score*	*Rank*
4661	38.77	59	4819	42.64	50	4838	41.22	56	4856	47.67	43
4801	50.77	33	4820	28.52	66	4839	50.25	36	4857	63.19	3
4802	53.06	26	4821	36	62	4840	58.47	11	4858	60.77	8
4803	53.72	25	4822	53.96	22	4841	62.24	5	4859	58.15	12
4805	52.80	28	4823	52.82	27	4842	53.95	23	4860	51.60	31
4806	52.16	30	4824	63.42	2	4843	70.81	1	4861	48.50	40
4807	51.42	32	4825	40.65	58	4844	41.58	55	4862	53.86	24
4808	35.55	63	4826	58.10	13	4845	37.22	60	4863	62.65	4
4809	34.10	65	4827	54.68	19	4846	41.99	54	4865	48.16	42
4810	54.09	21	4829	55.28	18	4847	42.24	53	4866	46.46	45
4811	48.51	39	4830	37.06	61	4848	61.23	7	4867	45.35	47
4813	42.25	52	4831	48.82	37	4849	56.38	15	4868	44.19	48
4814	56.08	17	4832	42.27	51	4850	58.69	9	4870	50.76	34
4815	56.49	14	4833	46.23	46	4851	41.12	57	4871	58.49	10
4816	52.75	29	4835	43.73	49	4852	48.59	38	4872	54.30	20
4817	34.46	64	4836	48.19	41	4854	56.22	16			
4818	61.55	6	4837	47.20	44	4855	50.74	35			

Table 16 *Score and rank after adjustment for absence from tests.*

Then we plot a distribution graph of the scores in Figure 22.

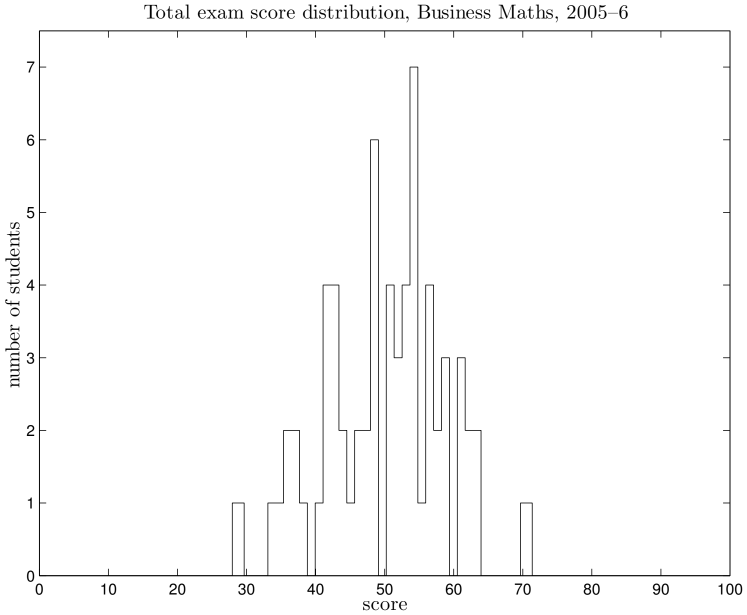


Figure 22 *Distribution of the total score of tests, attendance and homework.*

From the boundaries of each group portrayed in Figure 22 we arrive at the grading scheme in Table 17.

Range	*Grade*
$[66.81, 100]$	A
$[59.95, 66.81)$	B^+
$[49.66, 59.95)$	B
$[39.38, 49.66)$	C^+
$[31.38, 39.38)$	C
$[28, 31.38)$	D^+
$[0, 28)$	(N/A)

Table 17 *Grading scheme derived from Figure 22.*

And then the grades, together with total scores, are shown in Table 18.

ID	*Total*	*Grade*	*ID*	*Total*	*Grade*	*ID*	*Total*	*Grade*
4661	43.07	C^+	4824	63.42	B^+	4848	61.23	B^+
4801	50.77	B	4825	40.65	C^+	4849	56.38	B
4802	53.06	B	4826	58.10	B	4850	58.69	B
4803	53.72	B	4827	54.68	B	4851	41.12	C^+
4805	52.80	B	4829	55.28	B	4852	48.59	C^+
4806	52.16	B	4830	37.06	C	4854	56.22	B
4807	51.42	B	4831	48.82	C^+	4855	50.74	B
4808	35.55	C	4832	42.27	C^+	4856	47.67	C^+
4809	34.10	C	4833	46.23	C^+	4857	63.19	B^+
4810	54.09	B	4835	43.73	C^+	4858	60.77	B^+
4811	48.51	C^+	4836	48.19	C^+	4859	58.15	B
4813	42.25	C^+	4837	47.20	C^+	4860	51.60	C^+
4814	56.08	B	4838	41.22	C^+	4861	48.50	C^+
4815	56.49	B	4839	50.25	B	4862	53.86	B
4816	52.75	B	4840	58.47	B	4863	62.65	B^+
4817	34.46	C	4841	62.24	B^+	4865	48.16	C^+
4818	61.55	B^+	4842	53.95	B	4866	46.46	C^+
4819	42.64	C^+	4843	70.81	A	4867	45.35	C^+
4820	28.52	D^+	4844	41.58	C^+	4868	44.19	C^+
4821	36	C	4845	37.22	C	4870	50.76	B
4822	53.96	B	4846	41.99	C^+	4871	58.49	B
4823	52.82	B	4847	42.24	C^+	4872	54.30	B

Table 18 *Total score and grade, Business Mathematics, 2005–6.*

Here student 4860 who had been absent from too many exams was given a C^+ instead of a B. Also, she said she would come to Quiz 3, but did not. That makes her marks fairly $51.60(0.9) = 46.44$, that is a C^+.

Student 4661 had not come to Quiz 1. According to our procedure, therefore, the total of his score becomes $38.77/0.9 = 43.07$, and a C^+ at that.

Kit Tyabandha
Bangkok, 14th January, 2007

Judge not, that ye be not judged.
Matthew 7:1

Authors' profile

Kit Tyabandha

`arthur.tyabandha@web.de`

Education

2004	PhD	University of Manchester (UK)
1995	MSc	University of Manchester Institute of Science and Technology (UK)
1993	BEng	Electrical Engineering Chulalongkorn University (Thailand)
1992	BSc	Computer Science Ramkhamhaeng University (Thailand)
1991	BEng	Mineral Engineering Chulalongkorn University (Thailand)
1983	6th Form	Ashburton College (New Zealand) Certificate

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Kinder is a german word that means ‘children’. So the name of the place should more correctly be written ‘Kinderscout’. But then again there are Kinder Downfall and Kinder Low. To do the same thing everywhere would probably result in something seemingly out of place sitting in an English context. Therefore the name is normally written ‘Kinder Scout’.

Kit Tyabandha
Bangkok, April 2006